Analytic Calculation of Hall Reconnection

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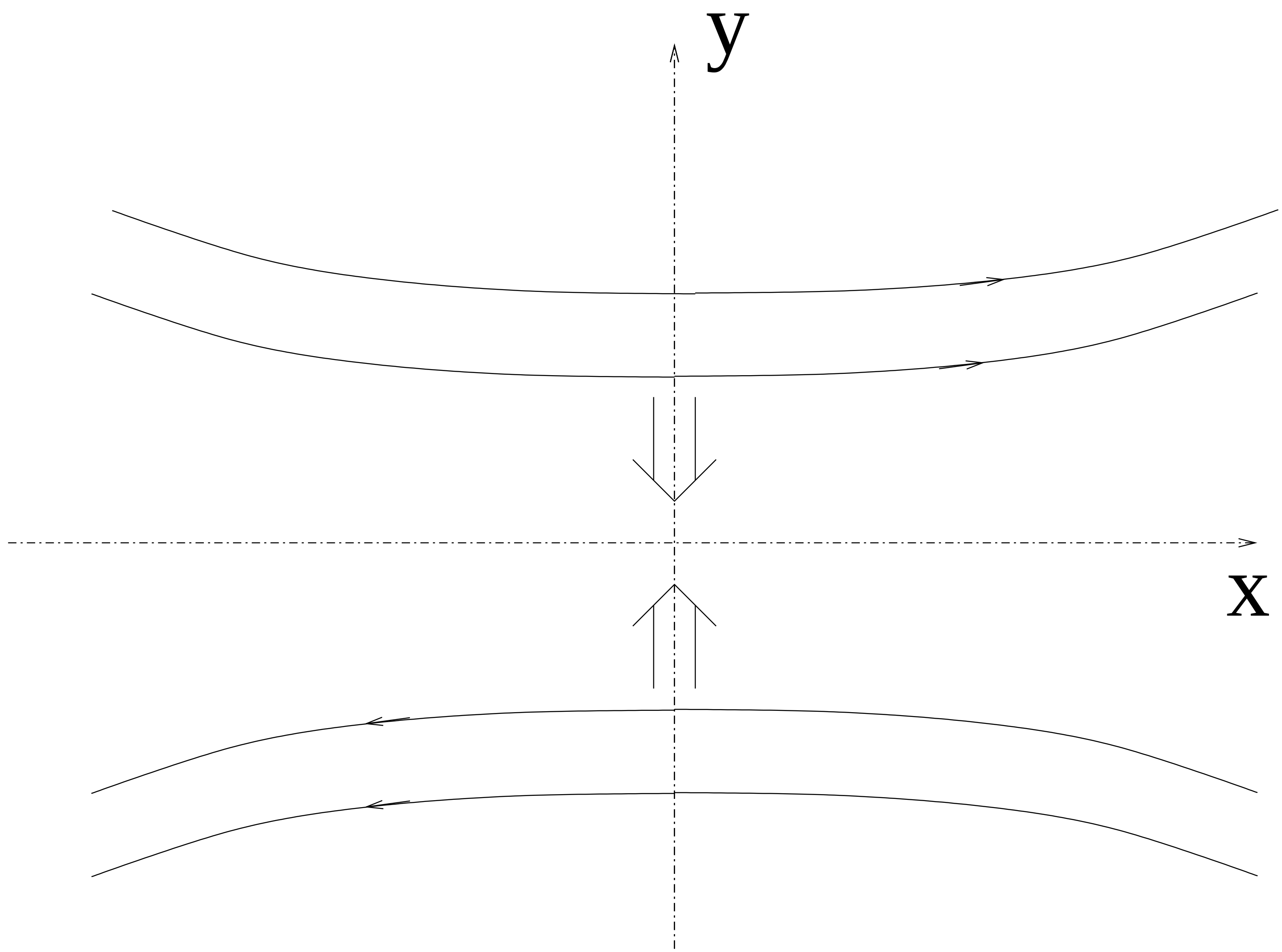
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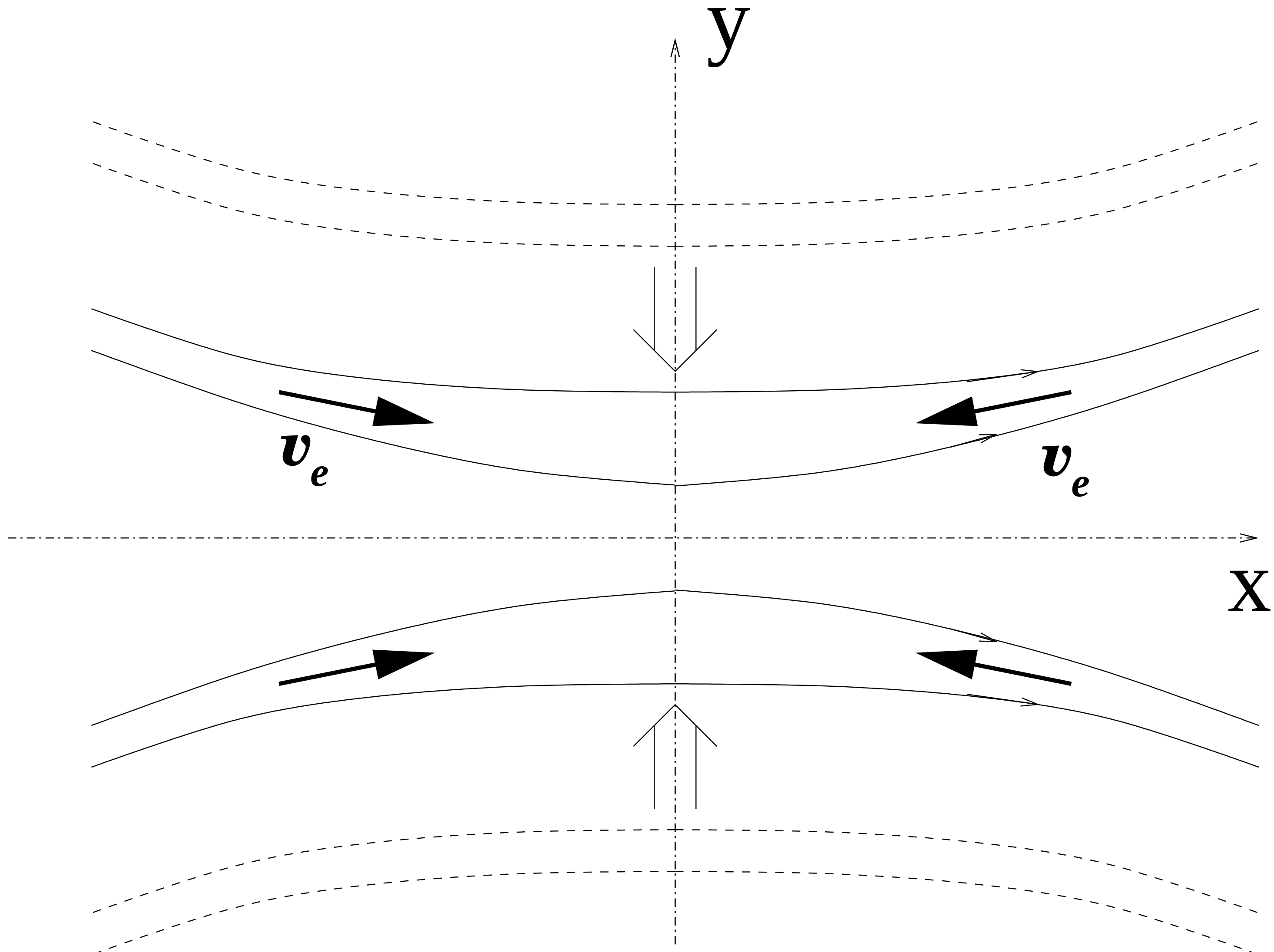

y
v_e
v_e
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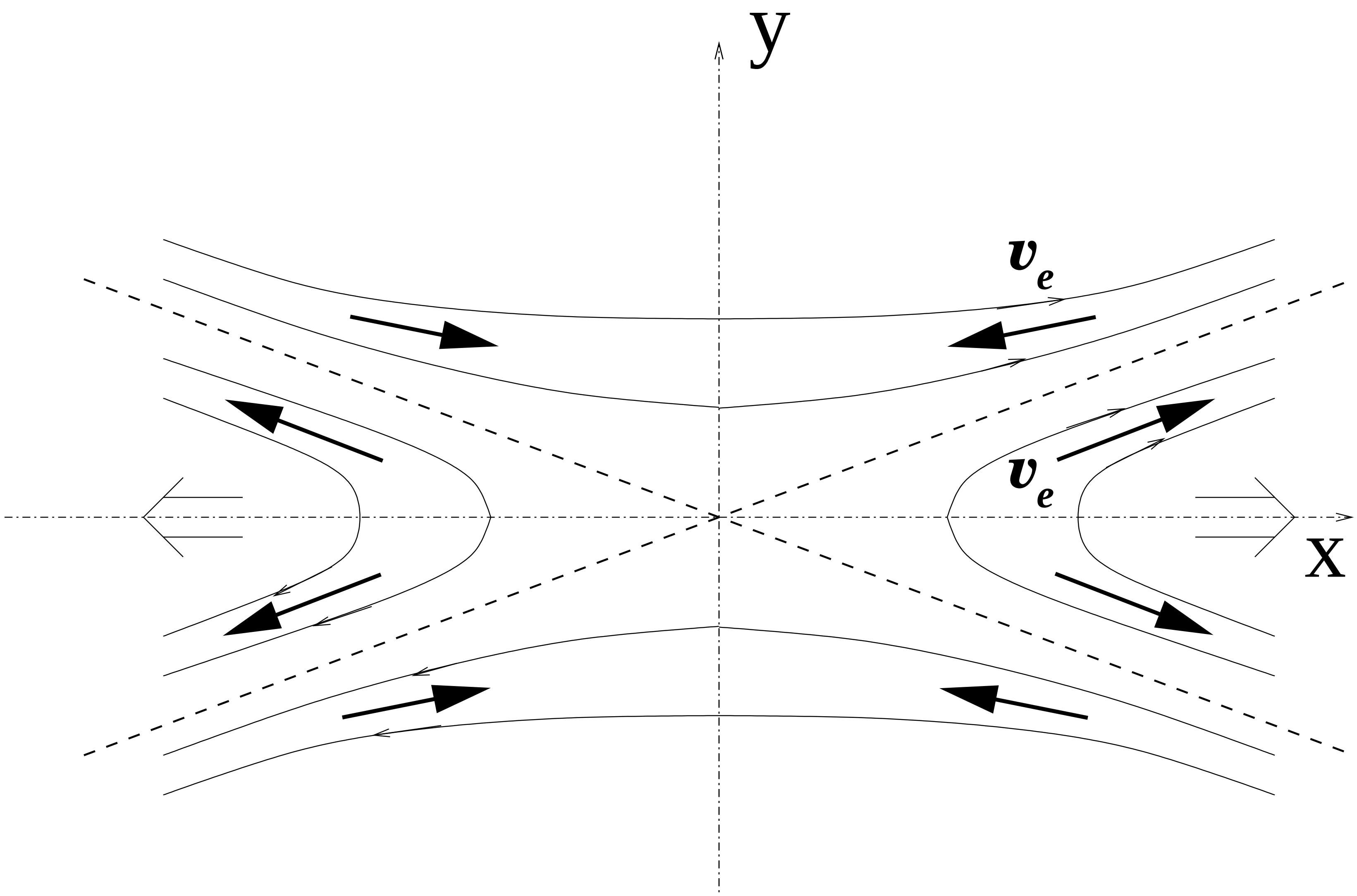
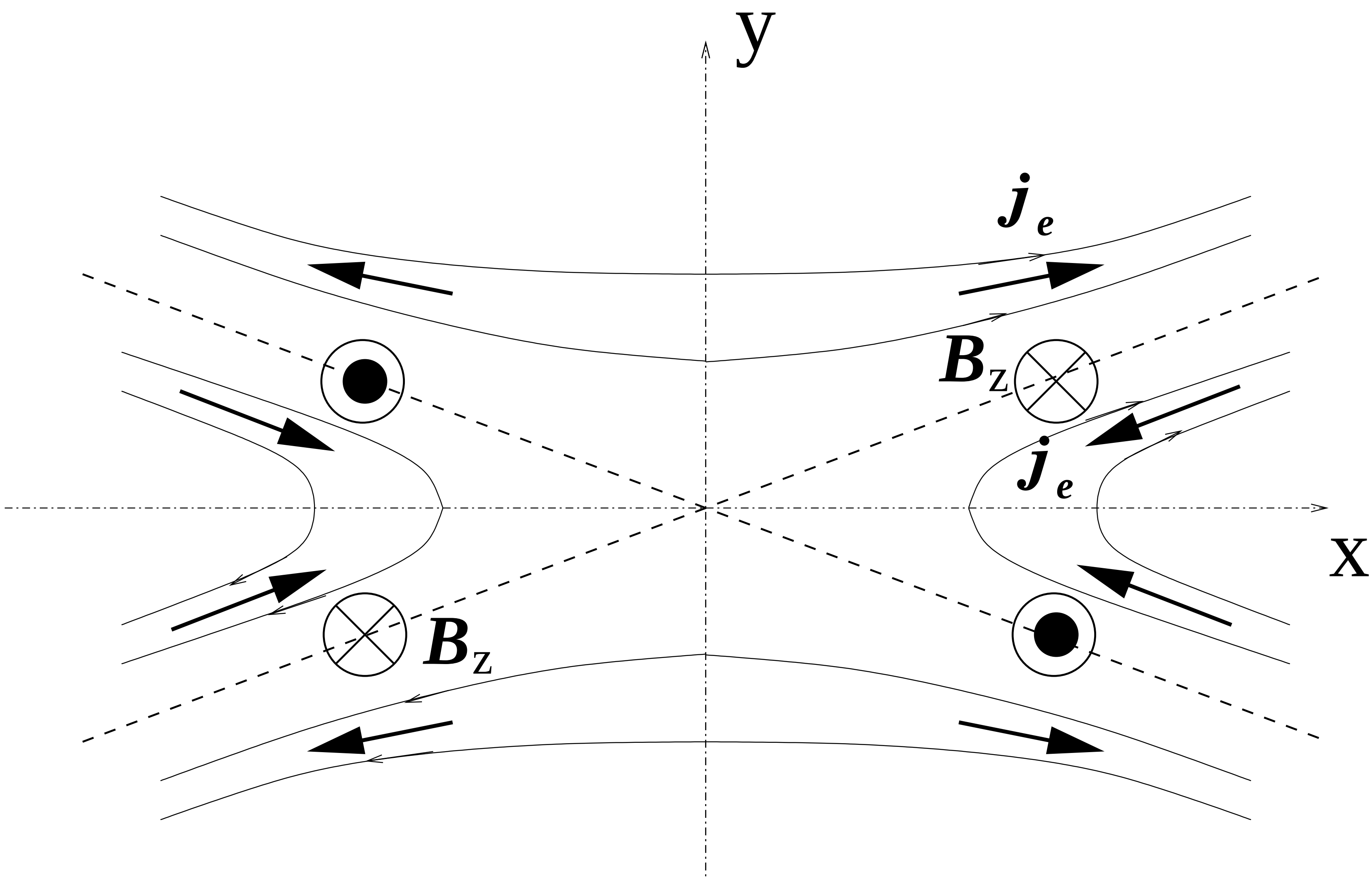


y
v_e
v_e
x



y
X
j_e
j_e
B_z
B_z

Ideal Incompressible Electron MHD: Assumptions

Assumptions:

- Translational symmetry in the z direction: $\partial_z = 0$.

- Steady state: $\partial_t = 0$.

  Then, $\mathbf{E} = -\nabla\phi = E_z\hat{z} - \nabla_\perp f(x, y)$
  with $E_z = \mathrm{const.}$

- Ideal electron MHD: $c\mathbf{E} = \mathbf{v}^{(e)} \times \mathbf{B}$.

- Incompressibility: $\mathbf{v}^{(e)}_{\mathrm{pol}} = \nabla\Phi_e \times \hat{\mathbf{z}}$.

- Ions provide uniform charge but no electric current: $\mathbf{j} \approx \mathbf{j}^{(e)}$.

Ideal Incompressible Electron MHD: General Results

Three Important Functions:

– volume per flux $V(x,\Psi) = \int_0^x \frac{dl_{\rm pol}}{B_{\rm pol}}|_\Psi$

– electron stream function Φ_e

– electron contribution to the toroidal field $B_z^{(e)}$

General Relationships between them:

- From incompressibility and flux freezing:

$$\Phi_e(x,\Psi) = c\,|E_z|\,V(x,\Psi)\,. \qquad (1)$$

- From Ampere's law:

$$B_z^{(e)} = -\,D\,\Phi_e = -\,cD|E_z|\,V(x,\Psi)\,, \qquad (2)$$

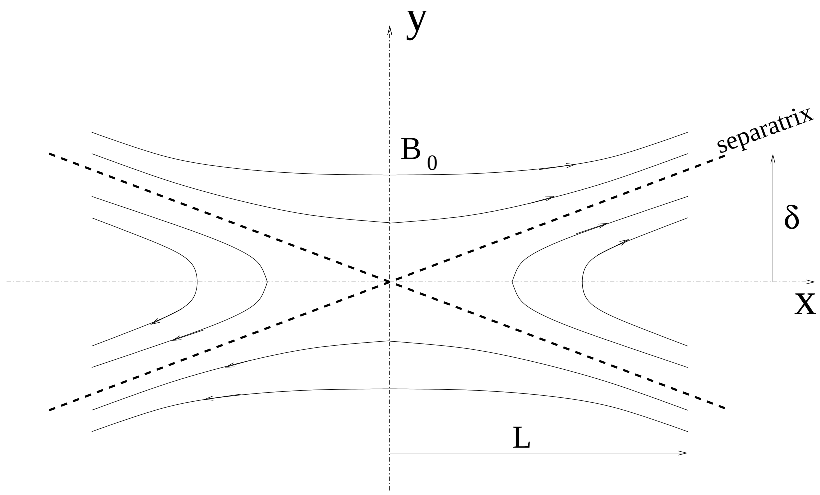
where $D \equiv 4\pi n_e e/c = B_0/(d_i V_A) = \text{const.}$

Eqn. (2) $\Rightarrow$ $\mathbf{v}_{\rm pol}^{(e)} \cdot \nabla B_z^{(e)} \equiv 0.$

But $(d/dt)\,B_z^{(e)} = \mathbf{v}_{\rm pol}^{(e)} \cdot \nabla B_z^{(e)} = \mathbf{B}_{\rm pol}^{(e)} \cdot \nabla v_z^{(e)}$.

Thus, $v_z^{(e)}$ and $j_z^{(e)}$ have to be constant along field lines: $\nabla^2\Psi = F(\Psi)$.

Example: Simple X-point Configuration



$$\bar{x} = \frac{x}{L}$$

$$\bar{y} = \frac{y}{\delta}$$

Simple X-point configuration: $\Psi(x,y) = \frac{B_0\delta}{2}\left(\frac{y^2}{\delta^2} - \frac{x^2}{L^2}\right)$ gives:

- Electron Velocity Field:

$$v_x^{(e)} = -\, x\, \frac{c\,|E_z|}{2\Psi(x,y)}$$

$$v_y^{(e)} = -\, y\, \frac{c\,|E_z|}{2\Psi(x,y)}$$

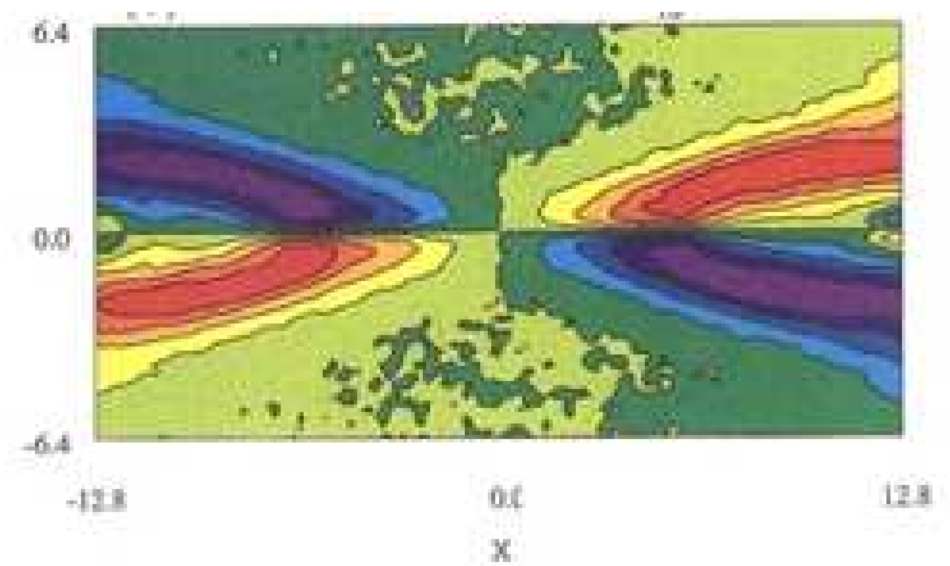
- Toroidal Magnetic Field:

$$B_z(x,y) = -\,\frac{B_0}{2}\frac{\delta}{d_i}\frac{u}{V_A}\,\log\left|\frac{y/\delta + x/L}{y/\delta - x/L}\right|.$$

- Main Features:
  - electron streamlines are straight radial rays $\bar{y} = C\bar{x}$;
  - B_z is simply advected by the electron fluid: $v_e \cdot \nabla B_z = 0$;
  - hence, $B_z = \mathrm{const}$ along rays $\bar{y} = C\bar{x}$;
  - B_z diverges logarithmically at the separatrix $\bar{y} = \bar{x}$.

Quadrupole Field in Numerical Simulations

Quadrupole Pattern of Toroidal Magnetic Field
seen in Numerical Simulations (2-fluid and kinetic):



Pritchett *et al.* 2001

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Field-Line Shape in xz Plane

Q: What is the shape $z(x, \Psi)$ of a field line in xz plane?

$$\frac{dz}{dx}\Big|_{\Psi} = \frac{B_z}{B_x}$$

Integrate:

$$\Delta z(x, \Psi) \equiv z(x, \Psi) - z(0, \Psi) \ = - \ c|E_z|D \ \frac{V^2(x, \Psi)}{2} \, .$$

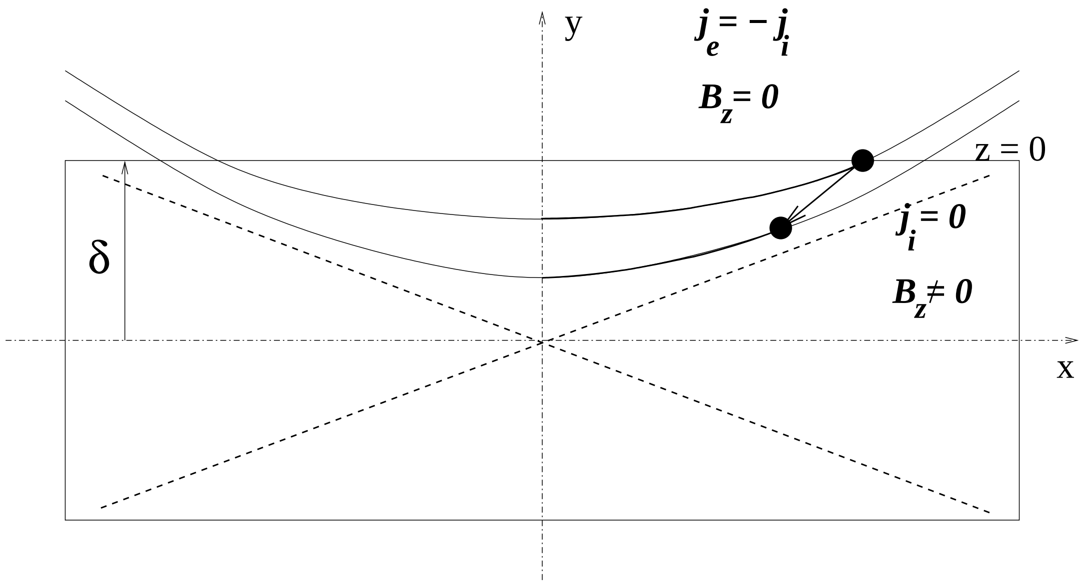
For a given e-fluid element with a trajectory $[X(t), \Psi(t)]$:

$$V[X(t), \Psi(t)] = \text{const} \quad \Rightarrow \quad \Delta z[X(t), \Psi(t)] = \text{const} \, .$$

The field line *looks* more and more stretched toroidally only because it is squeezed from the sides in the x direction, not because it is differentially stretched in the z direction!

B_z production: a simple illustration

Toroidal field is actually produced in the region where the ion flow is decoupling from the electron flow: $0 < |j_{\rm pol}^{(i)}| < |j_{\rm pol}^{(e)}|$.



B_z is produced at the sharp boundary $y = \delta$ and is then just advected by electron flow as a scalar:

- $\underline{y > \delta}$: $\quad j_{\rm pol} = j_{\rm pol}^{(e)} + j_{\rm pol}^{(i)} = 0 \Rightarrow B_z = 0.$
- $\underline{y = \delta}$: $\quad z(\delta, \Psi) \equiv 0.$
- $\underline{y < \delta}$: $\quad z(y, \Psi) = \Delta z(y, \Psi) - \Delta z(\delta, \Psi).$

For a given fluid element, $\quad \Delta z(Y, \Psi) \sim V^2(Y, \Psi) = \text{const}.$

Hence, any new length in the toroidal direction that is added to a field line as it moves deeper into the layer is just injected at $y = \delta$!

Bipolar Poloidal Electric Field

Q: *Why do field lines move in the toroidal direction?*

Electron eqn. of motion (Ohm's law) with isotropic electron pressure:

$$c\,E_y \simeq c\,E_{\mathrm{pol},\perp} = -v_z^{(e)} B_{\mathrm{pol}} - c\,\frac{\nabla_{\mathrm{pol},\perp} p_e}{n_e e}\,.$$

Two drifts: $\mathbf{E} \times \mathbf{B}$ and diamagnetic:

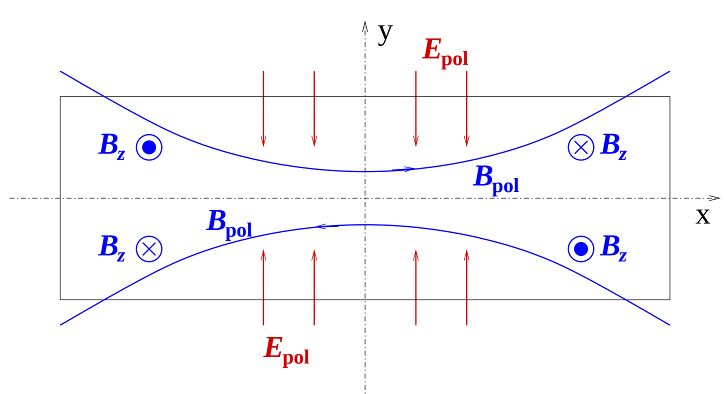
$$v_z^{(e)} = c\,\frac{[\mathbf{E}_{\mathrm{pol},\perp} \times \mathbf{B}_{\mathrm{pol}}]_z}{B_{\mathrm{pol}}^2} + \frac{[\nabla_{\mathrm{pol},\perp} p_e \times \mathbf{B}_{\mathrm{pol}}]_z}{B_{\mathrm{pol}}^2} = -\,c\,\frac{[\nabla_{\mathrm{pol},\perp} \tilde{\phi} \times \mathbf{B}_{\mathrm{pol}}]_z}{B_{\mathrm{pol}}^2}\,,$$

where $\tilde{\phi} \equiv \phi - p_e/n_e e$.

But the *field line velocity* v_B is just the $\mathbf{E} \times \mathbf{B}$ velocity:

$$v_{B,z} = c\,\frac{\mathbf{E}_{\mathrm{pol},\perp}}{B_{\mathrm{pol}}} \approx -\,c\,\frac{E_y}{B_x}\,.$$

The diamagnetic drift is irrelevant. Field lines move toroidally because of $\mathbf{E}_{\mathrm{pol},\perp}$ with a bipolar structure. This electric field is an important signature of Hall reconnection. It has been observed with spacecraft in Earth's magnetosphere (e.g., Mozer et al.).



INERTIAL EFFECTS

Ideal e-MHD: $\mathbf{v}_e$ blows up at the separatrix $\Psi = 0$.
Inertial terms become important and should resolve the singularity, leading to finite B_z.

E.g.:

$$z(Y, \Psi) = \Delta z(Y, \Psi) - \Delta z(\delta, \Psi) \sim -\left[V^2(Y, \Psi) - V^2(\delta, \Psi)\right] \Rightarrow$$

$$v_z^{(e)} = -\frac{d}{dt}\,\Delta z[\delta, \Psi(t)] \sim \frac{d}{dt}\,V^2[\delta, \Psi(t)] \sim E_z\,\frac{d}{d\Psi}\,V^2[\delta, \Psi] \sim (\log^2 \Psi)'$$

Inertial term:

$$m_e \mathbf{v}^{(e)}_{\mathrm{pol},\perp} \cdot \nabla_\perp\, v_z^{(e)} \simeq m_e E_z\,\frac{d}{d\Psi}\,v_z^{(e)} \sim e E_z \quad \Rightarrow$$

critical field line:

$$\Psi_{\mathrm{crit}} \sim \sqrt{\frac{m_e}{M}\sqrt{\log\frac{2M}{m_e}}} \simeq 0.04$$

Missing Pieces

What are we missing?

- Vertical Pressure Balance:

$$B_{\rm pol}^2 + B_z^2 + 8\pi P = B_0^2$$

Toroidal Field contributes to the pressure balance and hence affects the poloidal field structure.

- Breakdown of ideal electron MHD
(e.g., electron inertia, anisotropic pressure, resistivity):

For example,

$$\mathbf{v}_{\perp,{\rm pol}}^{\rm ideal} = c\,\frac{[\mathbf{E}\times\mathbf{B}]_{\rm pol}}{B^2} = c\,\frac{[\mathbf{E}\times\mathbf{B}_{\rm pol}]}{B_{\rm pol}^2}$$

is replaced by

$$\mathbf{v}_{\perp,{\rm pol}} = c\,\frac{[\mathbf{E}\times\mathbf{B}_{\rm pol}]}{B_{\rm pol}^2 + \varepsilon^2 B_0^2}$$

$\varepsilon \ll 1$

Magnetic Field along the Separatrix

- Toroidal field modifies the poloidal field structure through vertical pressure balance

- Along the separatrix $\bar{y} = \bar{x}$, the poloidal magnetic field satisfies ODE

$$-\frac{d}{d\bar{x}}\,\bar{B}_{z,s}(\bar{x}) = \frac{d}{d\bar{x}}\,(\bar{x}^2 - \bar{B}_{\mathrm{pol},s}^2)^{1/2} = Q\,\frac{\bar{B}_{\mathrm{pol},s}(\bar{x})}{\bar{B}_{\mathrm{pol},s}^2 + \varepsilon^2}$$

where
$$Q \equiv \frac{L v_{\mathrm{rec}}}{d_i V_A} = C\,\frac{\delta}{d_i}\frac{u}{V_A} \;\le 1$$

- For $\bar{x} \ll \bar{x}_{\max} \equiv 2\varepsilon^2/Q$:

$$\bar{B}_{\mathrm{pol},s}(\bar{x} \ll \varepsilon) \;\simeq\; \bar{x}$$
$$\bar{B}_{z,s}(\bar{x} \ll \varepsilon) \;\simeq\; -\frac{Q}{2\varepsilon^2}\,\bar{x}^2$$

- If $\varepsilon \ll Q$,

$$\bar{B}_{\mathrm{pol},s}(\bar{x} \gg \bar{x}_{\max}) \;\simeq\; \frac{\varepsilon^2}{Q} = \mathrm{const}$$
$$\bar{B}_{z,s}(\bar{x} \gg \bar{x}_{\max}) \;\simeq\; -\,\bar{x}$$

“Mandt–Drake Argument”: A Complimentary Approach

The “Mandt–Denton–Drake (1994) Argument”:

Quadrupole toroidal field is created via stretching of poloidal field lines by non-uniform toroidal electron velocity.

1. in ideal e-MHD magnetic field lines are frozen into electron fluid;

2. as a field line is advected into the layer, the electrons start to move in the z-direction to carry the layer’s current;

3. the central part of the field line moves faster and hence toroidal magnetic field is generated out of the poloidal field by the differential stretching:

$$\frac{d}{dt}\, B_z = \mathbf{v}_{\rm pol}^{(e)} \cdot \nabla B_z = \mathbf{B}_{\rm pol} \cdot \nabla v_z^{(e)}\,.$$