Thermal conduction in magnetic fields

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Why is calculation of thermal conduction in magnetic fields important for astrophysicists?

Hot cluster halo, $T \sim 5 \times 10^{7}\,\mathrm{K}$, $n \sim 2 \times 10^{-4}\,\mathrm{cm}^{-3}$

$\Downarrow$ Thermal conduction

Dense cluster core, $T \sim 2 \times 10^{7}\,\mathrm{K}$, $n \sim 2 \times 10^{-3}\,\mathrm{cm}^{-3}$

$\Downarrow$ X-rays

- Spitzer thermal conduction time in galaxy clusters is $t_{\mathrm{S}} \sim 3 \times 10^{8}\,\mathrm{yrs} \ll t_{\mathrm{H}} \sim 10^{10}\,\mathrm{yrs}$.
- Magnetic fields in clusters of galaxies $\Rightarrow$ lower thermal conduction.
- Cooling flows (exist only if heat conduction is low).
- Formation of clusters of galaxies and cooling catastrophe (heat conduction is too small $\Rightarrow$ a steep rise in the density of the relaxed cluster core).

$$\frac{Q_{\mathrm{halo}}}{t_{\mathrm{cond}}} \sim L_{\mathrm{X}}$$

$$t_{\mathrm{cond}} \gtrsim t_{\mathrm{H}} \quad \Rightarrow \quad \frac{\kappa_{\mathrm{eff}}}{\kappa_{\mathrm{S}}} \lesssim \frac{1}{30}$$

Why may thermal conduction be reduced in magnetic fields?

The electron gyro-radius is extremely small, so heat conducting **electrons travel along field lines.**

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While traveling along field lines, electrons become **trapped between magnetic mirrors**, therefore the conductivity parallel to magnetic field lines is reduced.	**Field lines are tangled, and long,** as a result, electrons travel longer distances between hot and cold regions of space ($\nabla_{\parallel} T$ is weaker).

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Reduction $\sim 1/2 - 1/5$	Reduction $1/3 - 1/300$ **???**

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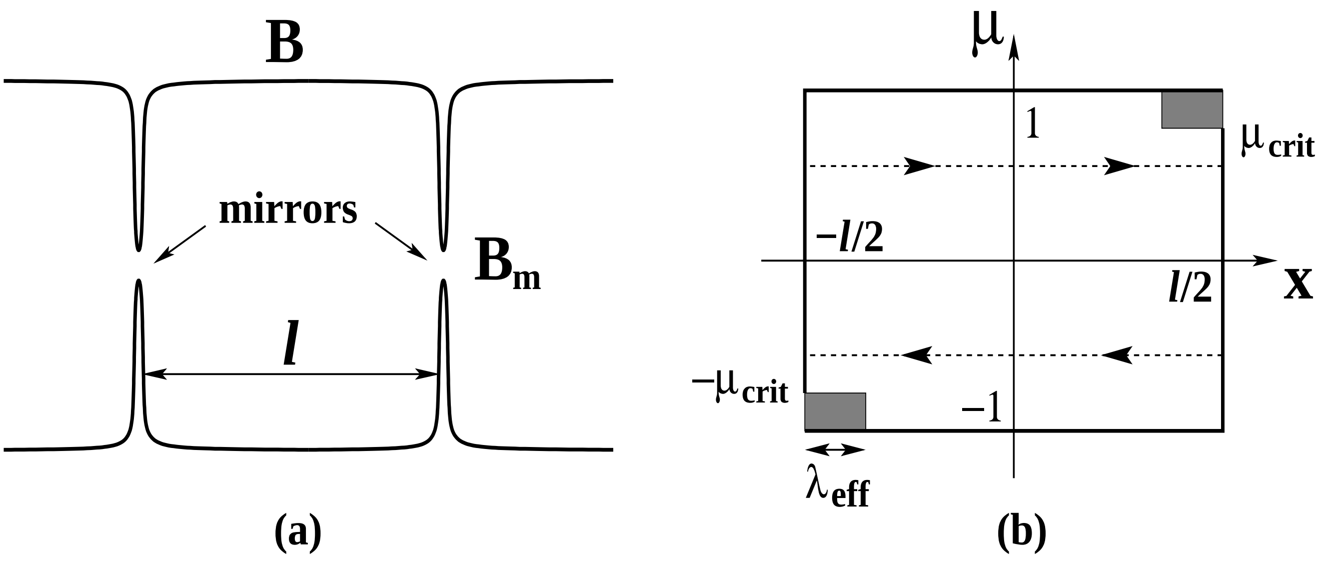
Total Reduction $1/5 - 1/2000$ **???**

Heat transport in magnetic mirrors

Escape of from a magnetic trap



Assume:

1. B is constant between the two mirrors;
2. mirror thicknesses is negligible compared to l;
3. mirror strength $m = B_{\rm m}/B$ is large;
4. all electrons are mono-energetic, $V = \rm const.$

The kinetic equation is

$$\frac{\partial f}{\partial t} + \mu V \frac{\partial f}{\partial x} = \frac{\nu}{2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial f}{\partial \mu}\right].$$

Electrons escape through two windows:

(1) $x = l/2$, $\mu > \mu_{\rm crit} = \sqrt{1-1/m} \approx 1 - 1/2m$, and

(2) $x = -l/2$, $\mu < -\mu_{\rm crit}$.

The electron distribution is in quasi-equilibrium

$$f(t,x) = e^{-t/\tau_m} F(x,\mu), \quad \tau_m \gg \nu^{-1}.$$

The solution of the kinetic equation and, therefore, the escape time τ_m, depends on mirror strength m and the ratio l/λ.

There are three limiting cases for which simple approximate solutions exist:

$$\begin{aligned}
&\tau_{m(1)} = \nu^{-1} \ln m, && l \ll \lambda_{\text{eff}}, \\
&\tau_{m(2)} = \nu^{-1}(l/\lambda_{\text{eff}}) = \nu^{-1}(2ml/\lambda), && \lambda_{\text{eff}} \ll l \ll \lambda^2/\lambda_{\text{eff}}, \\
&\tau_{m(3)} = \nu^{-1}(3/\pi^2)(l/\lambda)^2, && \lambda^2/\lambda_{\text{eff}} \ll l.
\end{aligned}$$

Here

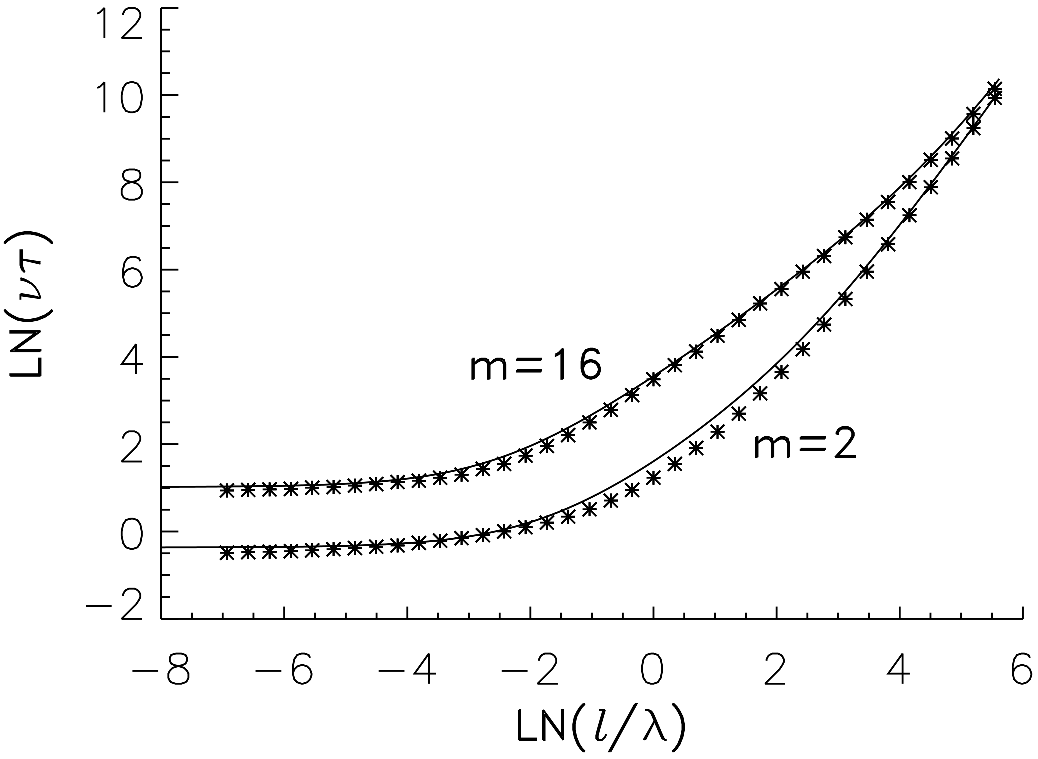
$$\lambda_{\text{eff}} = \lambda/2m \ll \lambda.$$

is the typical distance an electron travels in the loss cone $\mu > \mu_{\text{crit}}$ before it is scattered out of it is.

We use a simple interpolation formula

$$\begin{aligned}\tau_m &\approx \tau_{(1)} + \tau_{(2)} + \tau_{(3)} \\ &= \nu^{-1}\left[\ln m + (2ml/\lambda) + (3/\pi^2)(l/\lambda)^2\right]\end{aligned}$$

for the whole range of parameters m and l/λ.



Diffusion of mono-energetic electrons through a system of magnetic mirrors

- **The limit $\lambda \ll l_0$.**
  The escape time τ is controlled by the space diffusion along field lines. No reduction of diffusion.

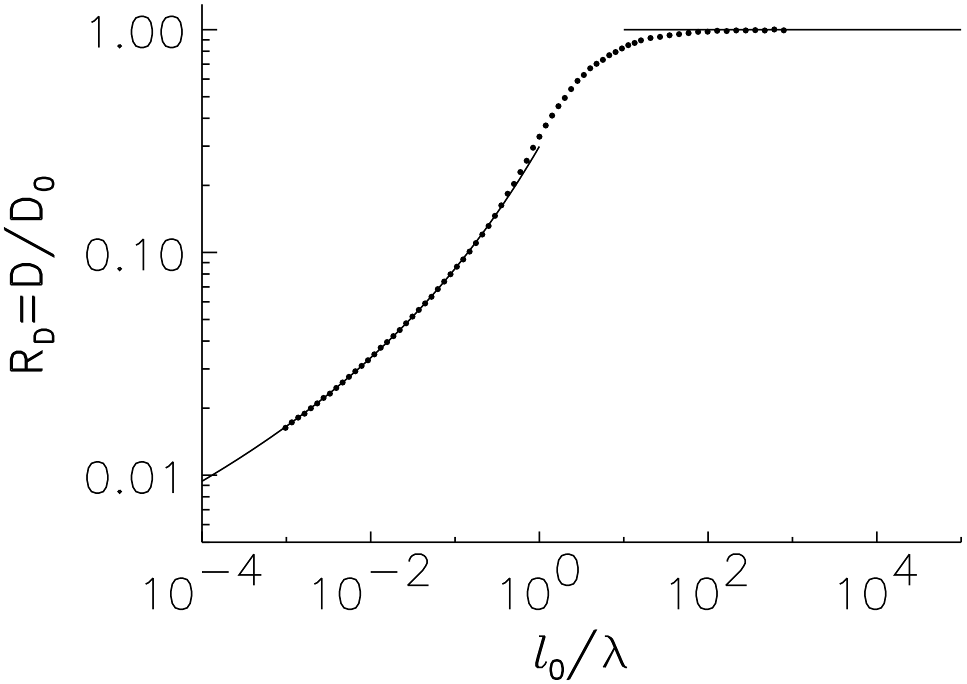
- **The limit $l_0 \ll \lambda$.**
  - Let the spectrum of magnetic mirrors be $\mathcal{P}(m)$.
  - Divide all mirrors into equal size bins $m \in (m - \delta/2, m + \delta/2)$. Each of the bins contributes to the inhibition of diffusion.
  - The bin that inhibits diffusion the most is the principle bin, it is $l_{m_p} \sim \lambda_{\text{eff}} \sim \lambda/m_p$.
  - For estimate of the diffusion reduction, we need to **consider only the magnetic mirrors that are in the principle bin**, neglecting the other bins.
  - As a result, electron diffusivity is

    $$D(l_0/\lambda) \sim \min_m \left\{C\left[l_m^2/\tau_m\right]\right\} \sim C\left[l_{m_p}^2/\tau_{m_p}\right].$$

  - We fit constants C and δ to our simulations.

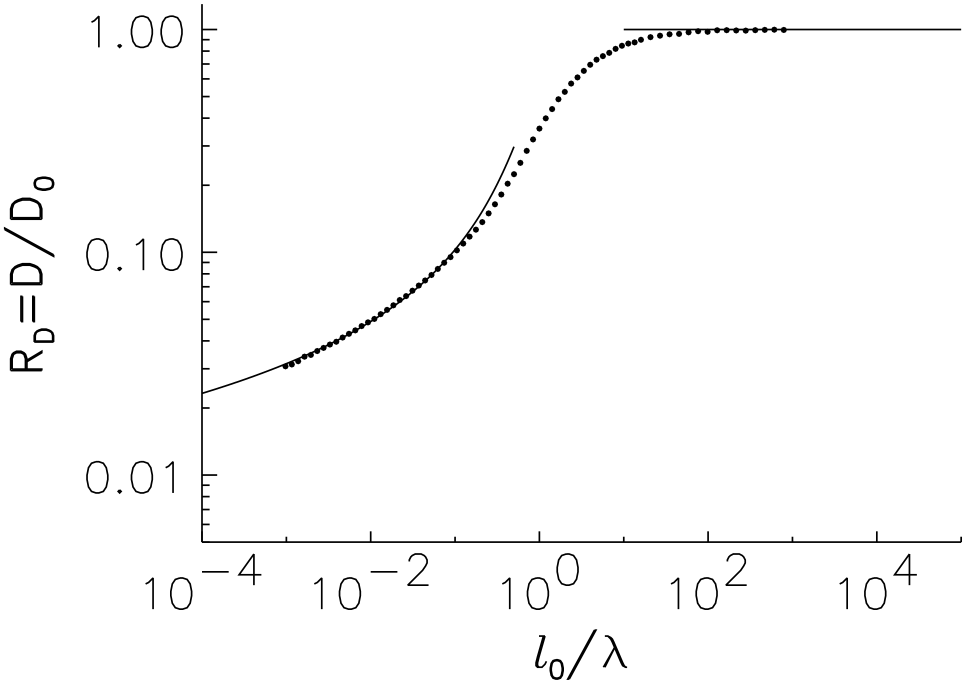
$$\mathcal{P}(m) = \exp\left[-(m-2)\right]$$

(a) Exponential: C=2.13, δ=2.60



$$\mathcal{P}(m) = (2/\pi)^{1/2}\exp\left[-(m-2)^2/2\right]$$

(b) Gaussian: C=1.68, δ=1.07



Spitzer kinetic equations

We combine the reduction of spatial diffusivity with the <u>full Spitzer kinetic equations</u>:

$$f(\mu, V) = f_0(V) + \mu n V_{\mathrm{T}}^{-3} S(v), \quad v = V/V_{\mathrm{T}},$$

$$\boxed{\hat{\mathcal{L}} S = \gamma_{\mathsf{T}}\, v^3 (2v^2 - 5) e^{-v^2} + \gamma_{\mathsf{E}}\, v^3 e^{-v^2} - \hat{\mathcal{I}} S,}$$

$$S(v) \to 0, \quad \text{as } \; v \to 0 \; \text{ and } \; v \to \infty,$$

$$\hat{\mathcal{L}} S(v) = \frac{d}{dv}\left[v G \frac{dS}{dv} \right] + 2v^2 G \frac{dS}{dv} - \left[\frac{1 + \Phi - G}{v R_{\mathrm{D}} (l_0/\lambda)} - 4v^2 \Phi' \right] S,$$

$$\hat{\mathcal{I}} S(v) = \frac{4}{15\sqrt{\pi}}\, e^{-v^2} \left[12 \overline{I}_5 - 10 \overline{I}_3 + 2v^3 (6v^2 - 5) \underline{I}_0 \right],$$

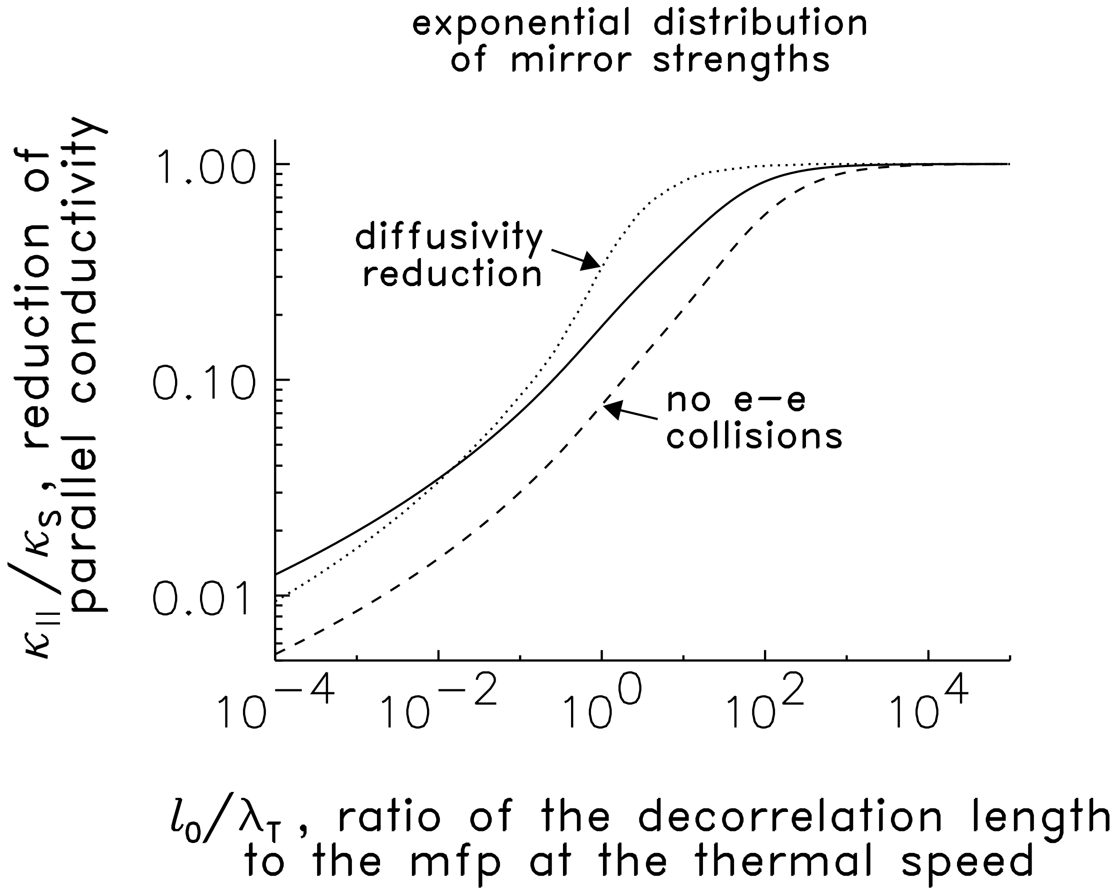
$$\overline{I}_m = \int_0^v v^m S(v)\, dv, \qquad \underline{I}_m = \int_v^\infty v^m S(v)\, dv,$$

$$\gamma_{\mathsf{T}} = \frac{k^2 T}{2\pi^{5/2} n e^4 \ln \Lambda} \frac{dT}{dx}, \qquad \gamma_{\mathsf{E}} = \frac{kT}{\pi^{5/2} n e^3 \ln \Lambda} E.$$

We solve these equations by relaxation technique and by making use of implicit numerical algorithms.

Parallel thermal conduction in magnetic mirrors

$$\left\{\begin{array}{rcl} j &=& \sigma E + \alpha\,(dT/dx) \;=\; 0, \\ Q &=& -\beta E - \kappa\,(dT/dx) \;=\; -\kappa_{\|}\,(dT/dx), \\ \kappa_{\|} &=& \kappa - \alpha\beta/\sigma \quad \underline{\text{parallel conductivity}}. \end{array}\right.$$



Effect of tangled field lines

Assume electrons are stuck to field lines

- Field lines, which connect cold and hot regions, are distributed in their length! There are short lines and there are long lines.
- Because electrons are stuck to field lines, all field lines are thermally isolated and conduct heat independently.
- As a result, there is no universal effective thermal conductivity. Effective conductivity depends on its definition in the context of the model we are solving!
- E.g., consider a toy-problem where the temperature in a galaxy cluster is constant at $t = 0$, and then it drops in time. Shorter lines cool off quickly, longer lines stay hot. Temperature becomes chaotic with position!
- Since most field lines are long, the diffusivity of electrons (and hence thermal conduction) is heavily suppressed:

$$D \sim D_0 (l_0/L)^2 \ll D_0.$$

Electrons are NOT stuck to field lines, assume single correlation scale field

- Field lines are tangled and therefore separate exponentially with the rate inversely proportional to the magnetic field scale l_0:

$$\frac{d\Delta}{dl} \sim \frac{\Delta}{l_0}, \qquad \text{if } \Delta < l_0$$

$$\Delta \sim \Delta_i \, e^{l/l_0}$$

- Electron drifts away of its field line by $\Delta_i = \rho_e$ each time it moves l_0 along the field line.
- Electron moves correlation length l_0 away and completely loses its initial field line, after traveling along field lines Rechester-Rosenbluth distance

$$L_{RR} \sim l_0 \, \ln \frac{l_0}{\rho_e}$$

- As a result, the diffusivity of electrons is

$$D \sim \frac{< r^2 >}{< t >} \sim \frac{(L_{RR}/l_0) l_0^2}{L_{RR}^2/D_{\|}} \sim \frac{D_0}{3\ln(l_0/\rho_e)} \sim \frac{D_0}{100}$$

Electrons are NOT stuck to field lines, assume multi-scale magnetic field

- Assume that 1D spectrum for magnetic field to be flatter than $k^{-2}\,dk$ (essentially a multi-scale field).
- As a result, equation for exponential separation of field lines now becomes

$$\frac{d\Delta}{dl} \sim \frac{\Delta}{\Delta}, \qquad \text{if } \Delta < l_0,$$

because magnetic fluctuations on scales $\sim \Delta$ are most efficient in separating field lines.

$$\Delta \sim l$$

- Electron moves correlation length l_0 (equal to the largest magnetic field scale) away and completely loses its initial field line, after traveling along field lines Rechester-Rosenbluth distance

$$L_{RR} = l_0$$

- As a result, the diffusivity of electrons is

$$D \sim \frac{< r^2 >}{< t >} \sim \frac{(L_{RR}/l_0) l_0^2}{L_{RR}^2/D_{\parallel}} \sim \frac{D_0}{3}$$

Questions to address and future work:

- What is the actual spectrum of magnetic mirrors? Is it steep? Can we measure it in experiments and/or simulations? Are strong mirrors thin?
- Can we compare different theoretical and numerical approaches and results on diffusion/conduction along magnetic field lines? Can diffusion along field lines and spatial wandering of lines be considered as separate effects?
- Can we put the problem of diffusion in multi-scale tangled field lines on a firm analytic (or numerical) footing? Note: so far all theoretical estimates have been qualitative, with exception of Jokipii 1973 (however $\delta B \ll B_0$ was assumed).
- Is the reduction of thermal conductivity in chaotic fields really equal to the reduction of electron diffusivity? Note that:
  (1) heat is transported by superthermal electrons;
  (2) in galaxy clusters

  $$\rho_e \ll \mathrm{mfp} < l_0 \ll L_{RR},$$

  so electrons collide a lot before going L_{RR}.

- Can we immediately use results or measurements from MHD turbulence problems in known transport theories? E.g. magnetic field spectrum?
- Can we check theories of diffusion and heat conduction in magnetic mirrors in experiments? What has been measured in MST? Can we use test particles in MST? Can we measure conduction in reconnection layer of MRX?
- Can we carry out full 3D simulations of heat transport, including collisions and e-e energy exchange? What is the best setup for such simulations (e.g., spreading of a temperature δ-function, stationary problem with temperature gradient and heat flux)?