2-D two-fluid MHD Reconnection
(work in progress)

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Outline

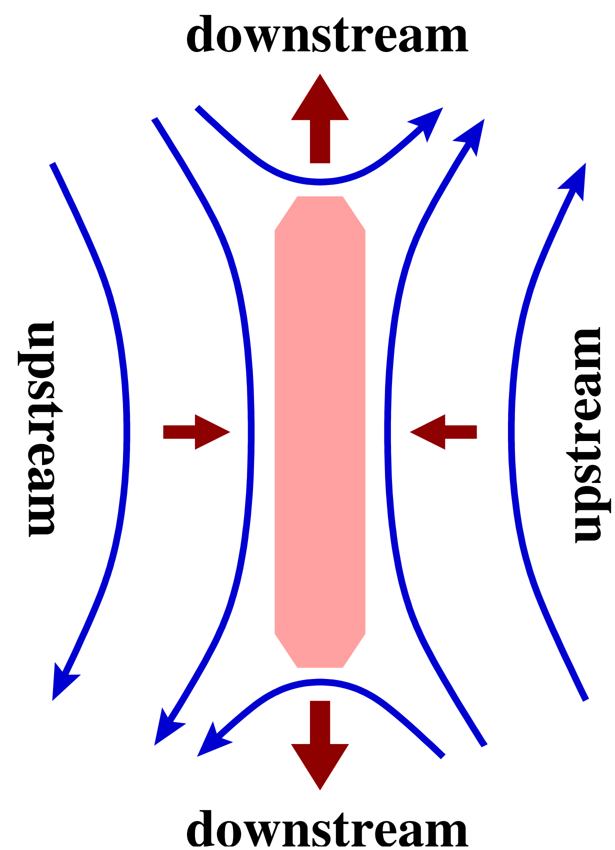
- Magnetic reconnection process
- Classical reconnection models
- The problem setup and our assumptions
- Single-fluid MHD equations
- Single-fluid MHD solution for reconnection with anomalous resistivity
- Two-fluid MHD equations
- Two-fluid MHD solution for reconnection
  (resistivity-independent collisionless reconnection)
- Summary

Magnetic reconnection process

Wikipedia: "Magnetic reconnection is the process whereby magnetic field lines from different magnetic domains are spliced to one another, changing the overall topology of a magnetic field"

- Single-fluid Magnetohydrodynamics (MHD)
  = Maxwell equations,
  equation of plasma motion with
  magnetic forces,
  Ohm's law

- Two-fluid MHD (collisionless)
  = Maxwell equations,
  updated equation of plasma motion,
  generalized Ohm's law



Classical single-fluid MHD models

Units: $c = 1$ and $4\pi = 1$

Energy conservation $\Rightarrow$

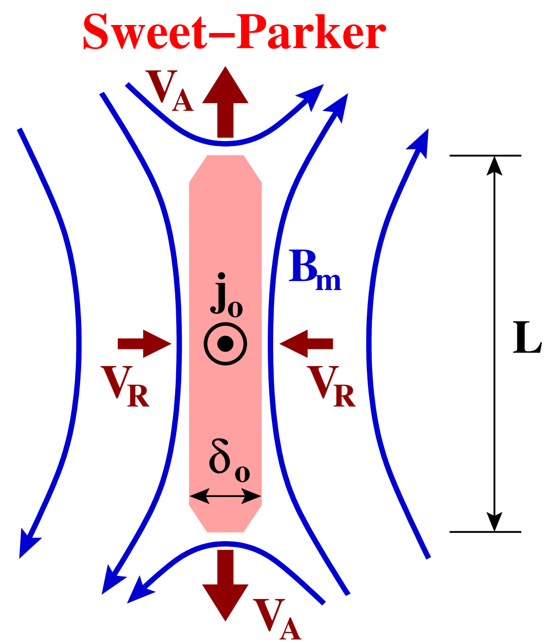
$$\eta_o j_o^2 \approx (V_R/\delta_o) B_m^2$$

Ampere's law $\Rightarrow$

$$j_o = (\nabla \times \mathbf{B})_z \approx B_m/\delta_o$$

Mass conservation $\Rightarrow$

$$V_R L' \approx V_A \delta_o$$

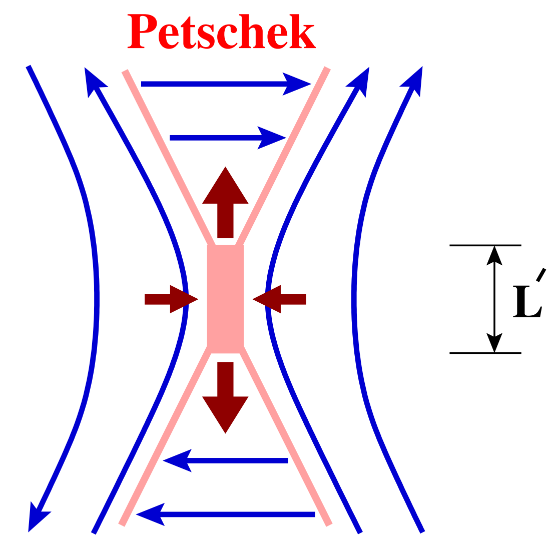


We obtain

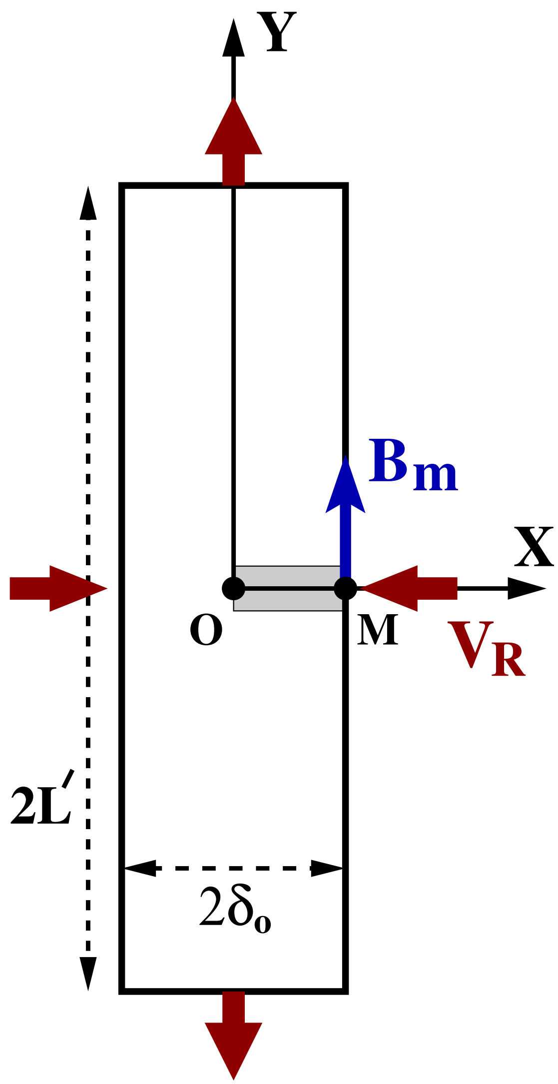
$$V_R/V_A \approx S_o^{-1/2}\sqrt{L/L'},$$

$$\text{where}\quad S_o \stackrel{\text{def}}{=} \sqrt{V_A L/\eta_o} \gg 1$$

$$V_R/V_A \approx \begin{cases} S_o^{-1/2}, & \text{Sweet-Parker},\ L' \approx L; \\ (\ln S_o)^{-1}, & \text{Petschek},\ L' \approx (L/S_o)(\ln S_o)^2 \end{cases}$$



The problem setup & assumptions



We assume

- **full** non-relativistic two-fluid MHD;
- 2-dimensional, $\partial/\partial z \equiv 0$;
- symmetries of reconnection layer;
- large Lundquist $\Leftrightarrow$ negligible η outside;
- quasi-stationarity $\Leftrightarrow$ slow rate $V_R/V_A \ll 1$;
- no plasma instabilities;
- thin reconnection layer $\delta_o/L' \ll 1$.
- neglect viscosity;
- isotropic pressure *(tensor in the future)*

Additional assumptions (for simplicity)

- constant resistivity η for two-fluid MHD;
- plasma incompressibility div $\mathbf{V} = 0$;
- zero guide field $B_z \equiv 0$ (will add quadrupole);

Single-fluid MHD: Equations

Ampere's Law without the displacement current:

$$\mathbf{j} = \nabla \times \mathbf{B} \quad \Rightarrow \quad j_z = \partial B_y/\partial x - \partial B_x/\partial y \approx B_m/\delta_o \quad \text{at O-point}$$

Faraday's Law, Quasi-stationarity $\partial/\partial t \approx 0$, 2-dimensional $\partial/\partial z \equiv 0$:

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \approx 0 \quad \Rightarrow \quad E_z \approx E_z(t) = \text{constant in space}$$

Incompressible plasma (also true for compressible plasma at O-point):

$$\partial_y V_y = -\partial_x V_x \approx V_R/\delta_0 \quad \text{at O-point}$$

Equation of plasma motion, $\partial/\partial t \approx 0$, $\partial/\partial z \equiv 0$:

$$\rho(\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla P + \mathbf{j} \times \mathbf{B} \quad \Rightarrow$$

$$\rho(\mathbf{V} \cdot \nabla)V_y = -\partial_y P + j_z B_x - j_x B_z = -\partial_y(P + B_z^2/2) + j_z B_x$$

Ohm's Law:

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{j} \quad \Rightarrow \quad E_z = -V_x B_y + V_y B_x + \eta j_z$$

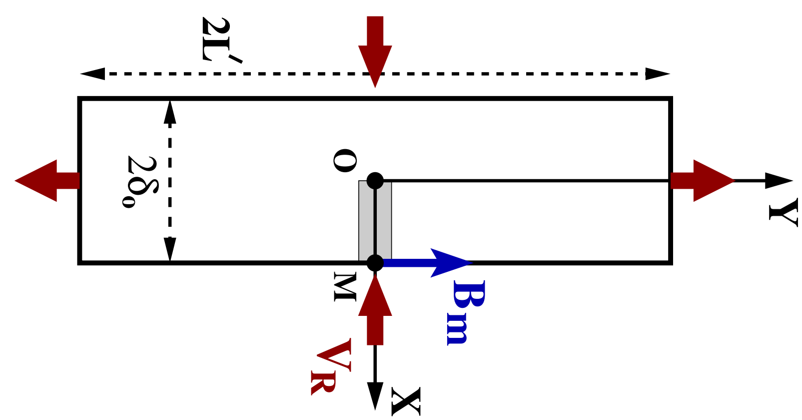
Single-fluid MHD: Global vs Local

	Global eqs.	**Local eqs.**
Ampere's law	$j_o = (j_z)_o = (\partial_x B_y)_o \approx B_m/\delta_o$	
Incompress.	$V_R L' \approx V'_{\rm out}\delta_o$	$(\partial_y V_y)_o = -(\partial_x V_x)_o \approx V_R/\delta_o$
Acceleration	$V'_{\rm out} \approx V_A$	$\rho(\partial_y V_y)_o^{\,2} = -\partial_y^2\left(B_y^2/2\right)_m + j_o(\partial_y B_x)_o$
Shocks	$B'_x/\sqrt{\rho} = V'_R \approx V_R$	
const E_z across	$\eta j_o = V_R B_m$	
const E_z along	$\eta j_o = \eta' j'_z + V'_{\rm out} B'_x$	$-\partial_y^2\left(\eta j_z\right)_o = 2(\partial_y V_y)_o(\partial_y B_x)_o$
Unknown	$j_o\ \delta_o\ V_R\ L'\ V'_{\rm out}\ B'_x$	$j_o\ \delta_o\ V_R\ (\partial_y V_y)_o\ (\partial_y B_x)_o$

$$\eta j_z = \eta_o j_o + (y^2/2)\{j_o(\partial_y^2\eta)_o + \\ + [\eta_o + j_o(\partial_{j_z}\eta)_o](\partial_y^2 j_z)_o\},$$

$$\partial_y^2\left(B_y^2/2\right)_m = B_m(\partial_y^2 B_y)_m \overset{\rm def}{=} -2B_m^2/L^2,$$

$$j_o^{-1}(\partial_y^2 j_z)_o \approx B_m^{-1}(\partial_y^2 B_y)_m = -2/L^2.$$



Single-fluid MHD: Solution

Scales: $L^2 \equiv -\frac{2B_m}{(\partial^2 B_y/\partial y^2)_m}, \qquad l_\eta^2 \equiv -\frac{2\eta_o}{(\partial^2 \eta/\partial y^2)_o},$

- $\eta = \text{const} = \eta_o \quad \Rightarrow$
  Sweet-Parker reconnection $V_R/V_A \approx V_A/\sqrt{S}$, where $S = V_A L/\eta_o$,
  with constant resistivity reconnection is Sweet-Parker, not Petschek.

- $(j_o/\eta_o)(\partial\eta/\partial j_z)_o \gg 1$ and $L^2/l_\eta^2 \quad \Rightarrow$
  Petschek-Kulsrud reconnection $V_R/V_A \approx \left[(B_m/V_A L^2)(\partial\eta/\partial j_z)_o\right]^{1/3}$,
  anomalous resistivity reconnection is much faster than Sweet-Parker.

- $L^2/l_\eta^2 \gg 1$ and $(j_o/\eta_o)(\partial\eta/\partial j_z)_o \quad \Rightarrow$
  localized resistivity case $V_R/V_A \approx V_A/\sqrt{S^*}$, where $S^* = V_A l_\eta/\eta_o$.

Two-fluid MHD: Equations

Ampere's Law without the displacement current:

$$\mathbf{j} = \nabla \times \mathbf{B} \quad \Rightarrow \quad j_z = \partial B_y/\partial x - \partial B_x/\partial y \quad \Rightarrow$$

$$j_o = (j_z)_o \approx B_m/\delta_o \quad \textbf{at O-point}$$

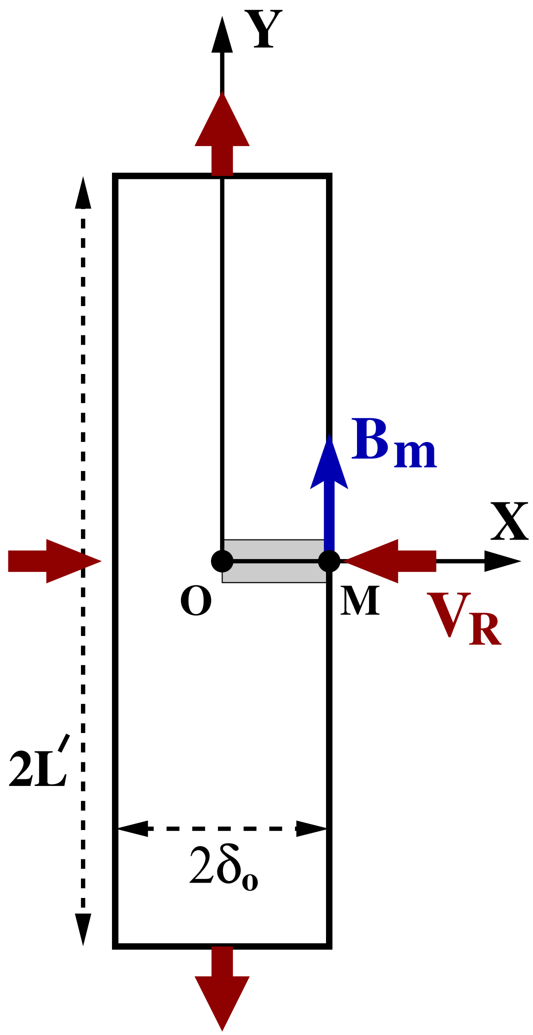
Faraday's Law, $\partial/\partial t \approx 0$, $\partial/\partial z \equiv 0$:

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \approx 0 \quad \Rightarrow$$

$$E_z \approx E_z(t) = \textbf{constant in space}$$

Incompressible plasma (also true for compressible plasma at O-point):

$$\partial_y V_y = -\partial_x V_x \approx V_R/\delta_0 \quad \textbf{at O-point}$$



Two-fluid MHD: Equations (continue)

Pressure tensor $P_{ij} = \langle (v_i - u_i)(v_j - u_j) \rangle$, where $\mathbf{u} = \langle v \rangle$

Equation of plasma motion:

$$\rho(\partial \mathbf{V}/\partial t) + \rho(\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla \cdot P + \mathbf{j} \times \mathbf{B} - \frac{m_p m_e}{e^2} \nabla \cdot (\mathbf{jj}/\rho)$$

Ohm's Law:

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \eta \mathbf{j} + \frac{m_p}{\rho e} \mathbf{j} \times \mathbf{B} - \frac{m_p}{\rho e} \nabla \cdot P_e + \frac{m_e}{\rho e} \nabla \cdot P_p + \frac{m_p m_e}{\rho e^2} \Big[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot \Big(\mathbf{Vj} + \mathbf{jV} - \frac{m_p}{e \rho} \mathbf{jj} \Big) \Big].$$

Pressure tensor $P_{ij} = \langle (v_i - V_i)(v_j - V_j) \rangle$, $\mathbf{V} = (m_p n_p \mathbf{u_p} + m_e n_e \mathbf{u_e})/\rho$

Equation of plasma motion:

$$\rho(\partial \mathbf{V}/\partial t) + \rho(\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla \cdot P + \mathbf{j} \times \mathbf{B}$$

Ohm's Law:

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \eta \mathbf{j} + \frac{m_p}{\rho e} \mathbf{j} \times \mathbf{B} - \frac{m_p}{\rho e} \nabla \cdot P_e + \frac{m_e}{\rho e} \nabla \cdot P_p + \frac{m_p m_e}{\rho e^2} \Big[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot \Big(\mathbf{Vj} + \mathbf{jV} \Big) \Big].$$

Two-fluid MHD: Equations (continue)

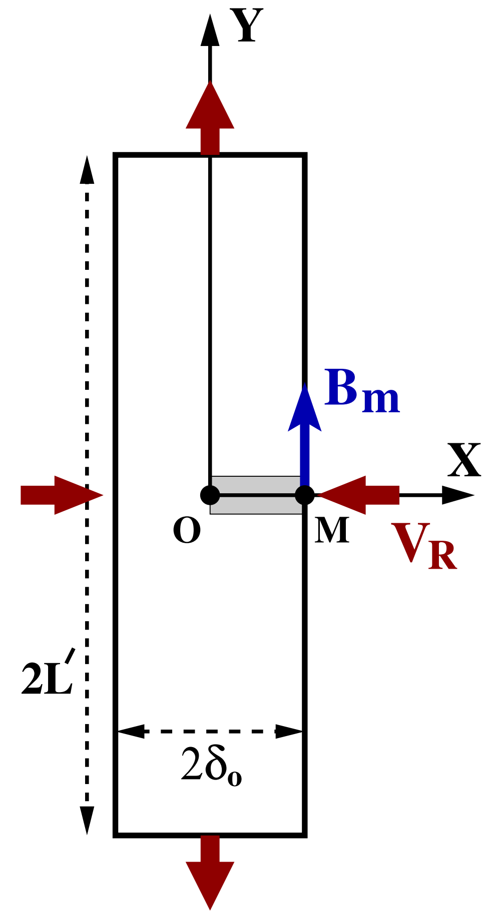
y-component of eq. of plasma motion, $\partial/\partial t \equiv 0$, $\partial/\partial z \equiv 0$, $B_z \equiv 0$:

$$\rho(\mathbf{V}\cdot\nabla)V_y = -\partial_y P + j_z B_x,$$

Take $\partial/\partial y$ at O-point $\Rightarrow$

$$\rho(\partial_y V_y)_o{}^2 = -(\partial_y^2 P)_o + j_o(\partial_y B_x)_o$$
$$= -B_m(\partial_y^2 B_y)_m + j_o(\partial_y B_x)_o$$
$$= 2B_m^2/L^2 + j_o(\partial_y B_x)_o,$$

where $L^2 \equiv -2B_m\,/(\partial_y^2 B_y)_m$



Two-fluid MHD: Equations (continue)

z-component of Ohm's Law, $\partial/\partial t \equiv 0$, $\partial/\partial z \equiv 0$, $B_z \equiv 0$:

$$\text{const} \approx E_z = -V_x B_y + V_y B_x + \eta j_z + d_e^2 (V_x \partial_x j_z + V_y \partial_y j_z), \qquad d_e^2 = \frac{m_p m_e}{\rho e^2}.$$

- O-point: $E_z = \eta (j_z)_o \equiv \eta j_o$

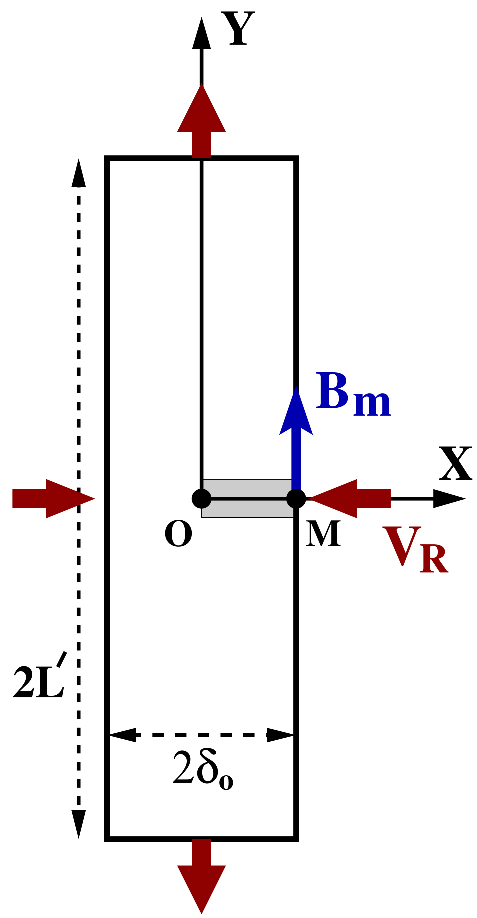
- M-point:
  $E_z = V_R B_m + \eta j_m - d_e^2 V_R (\partial_x j_z)_m = V_R B_m$

- $\partial^2 E_z / \partial y^2$ derivative at O-point
  (**E_z = const along layer**):

  $$0 = 2(\partial_y V_y)_o (\partial_y B_x)_o + \left[\eta + 2d_e^2 (\partial_y V_y)_o\right] (\partial_y^2 j_z)_o$$

  $$\approx 2(\partial_y V_y)_o (\partial_y B_x)_o - \left[\eta + 2d_e^2 (\partial_y V_y)_o\right] \frac{2 j_o}{L^2},$$

  where $(\partial_y^2 j_z)_o / j_o = -2/L^2$



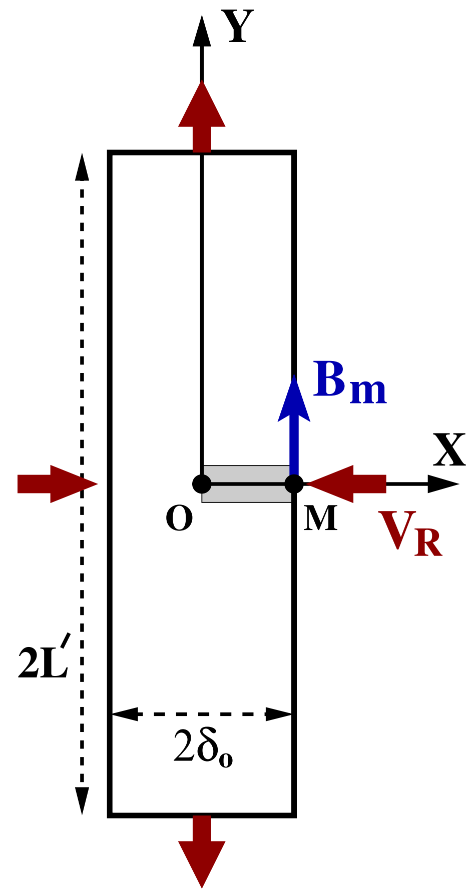
Two-fluid MHD: Equations (summary)

Ampere's Law: $j_o \equiv (j_z)_o \approx B_m/\delta_o$ **at O-point**

Faraday's Law: $E_z \approx$ **constant in space**

Incompressible at O-point:

$$\partial_y V_y = -\partial_x V_x \approx V_R/\delta_0 \quad \textbf{at O-point}$$



y-component of equation of plasma motion:

$$\rho(\partial_y V_y)_o{}^2 = \frac{2B_m^2}{L^2} + j_o(\partial_y B_x)_o, \quad L^2 \equiv \frac{-2B_m}{(\partial_y^2 B_y)_m}$$

z-component of Ohm's Law, $E_z \approx$ const across & along:

$$\eta j_o = V_R B_m, \quad 0 = 2(\partial_y V_y)_o(\partial_y B_x)_o - \left[\eta + 2d_e^2(\partial_y V_y)_o\right]\frac{2j_o}{L^2}$$

Unknowns: j_o, δ_o, V_R, $(\partial_y V_y)_o$ and $(\partial_y B_x)_o$.

Two-fluid MHD: Solution with $B_z = 0$

General solution:

$$V_R = V_A \left(\frac{d_e^2}{L^2} + \sqrt{\frac{d_e^4}{L^4} + \frac{3}{S^2}} \right)^{1/2}, \qquad L^2 \equiv \frac{-2B_m}{(\partial_y^2 B_y)_m},$$

$$\delta_o = L \left(\frac{S^2 d_e^2}{L^2} + \sqrt{\frac{S^4 d_e^4}{L^4} + 3S^2} \right)^{-1/2}, \qquad S \equiv \frac{V_A L}{\eta}.$$

Sweet-Parker limit: $d_e \ll L/\sqrt{S} = \delta_{\rm SP}$

$$V_R \approx \frac{V_A}{\sqrt{S}}, \qquad \delta_o \approx \frac{L}{\sqrt{S}}.$$

Collisionless limit: $d_e \gg L/\sqrt{S} = \delta_{\rm SP}$

$$V_R = V_A \frac{\sqrt{2} d_e}{L}, \qquad \delta_o = L \frac{L}{\sqrt{2} S d_e},$$

resistivity-independent reconnection rate!!!

Summary

- Local analytical derivations: A quasi-stationary reconnection rate is fully determined by functional form of anomalous resistivity at the reconnection center and by field configuration in the upstream region!!!
- 1-fluid, $\eta = \text{const}$ $\Rightarrow$ Sweet-Parker $\eta j_o \approx B_m V_A/\sqrt{S} \propto \rho^{-1/4} T^{-3/4}$
- Resistivity is anomalous $\Rightarrow$ reconnection can be faster.
- Collisionless with zero guide field, $d_e/\delta_{\text{SP}} \propto (T/\rho)^{3/4} \gg 1$ $\Rightarrow$ reconnection rate is resistivity-independent, $\eta j_o \approx B_m V_A (d_e/L) \propto 1/\rho$.

  Joule heating $\Rightarrow$ $T\uparrow$ and $\rho\downarrow$ $\Rightarrow$ reconnection is more and more collisionless, faster and faster $\Rightarrow$ run-away reconnection process (solar corona, Earth's magnetosphere?).
- need and will add quadrupole guide field into the analysis (work in progress).