Dynamic Alignment and Spectrum of MHD Turbulence

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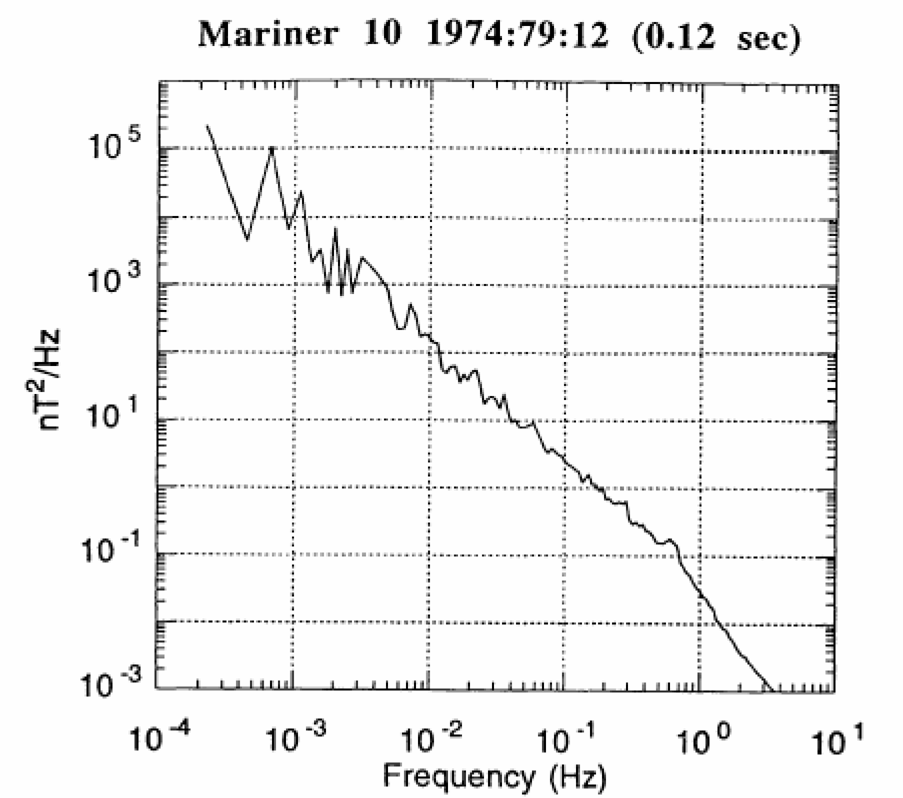
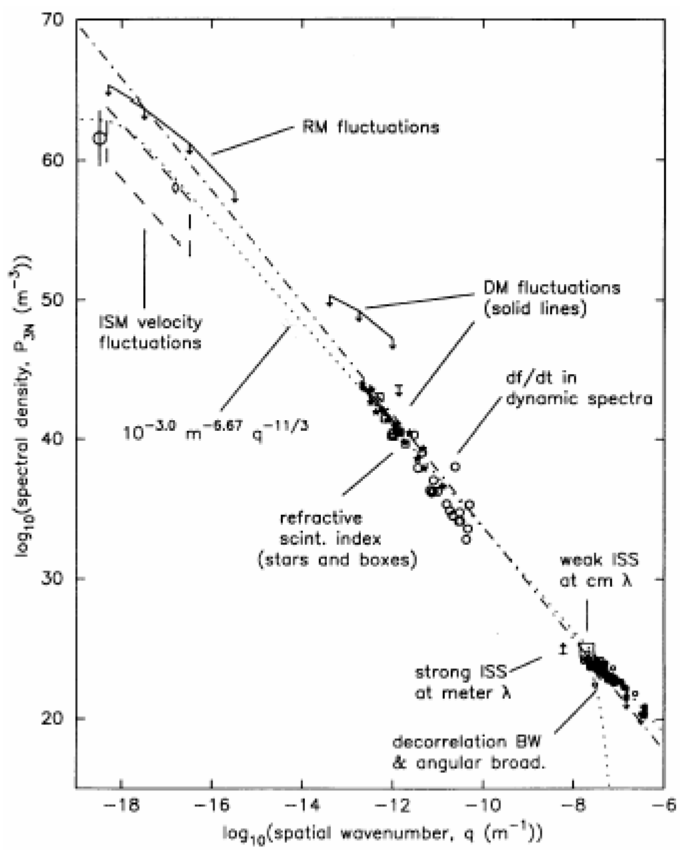
Observations



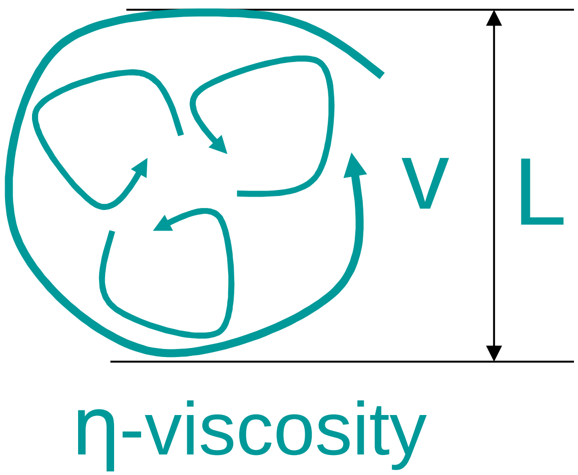
Solar wind



ISM

[Goldstein, Roberts, Matthaeus (1995)] [Armstrong, Rickett, Spangler (1995)]

Introduction: Kolmogorov turbulence



Random flow of incompressible fluid

Reynolds number:

Re=Lv/η>>1

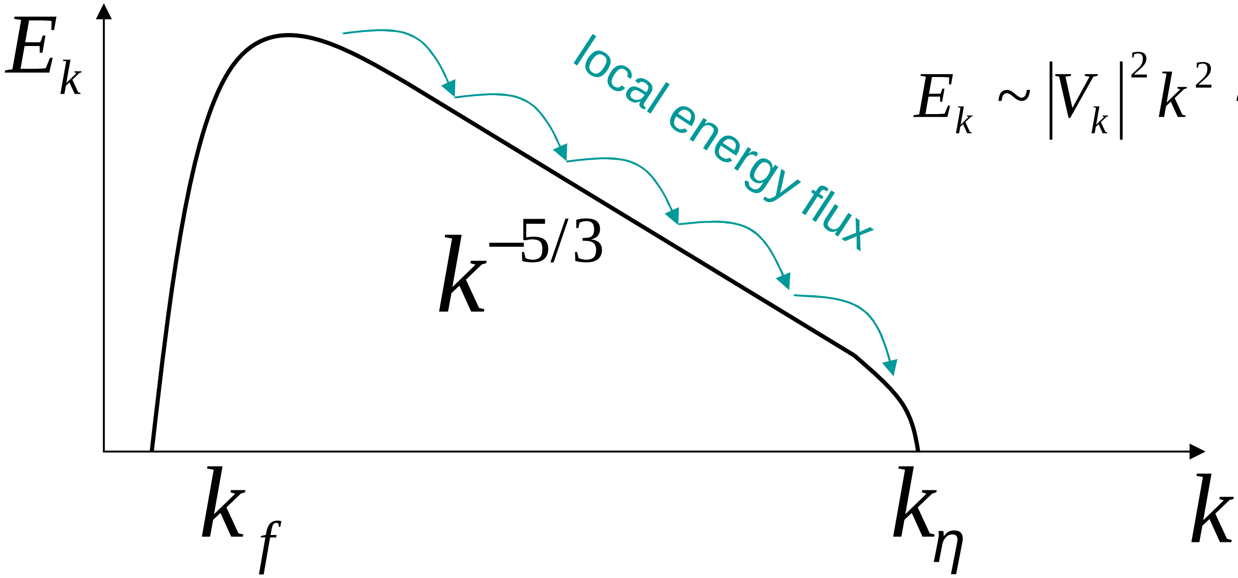
$$\left\langle \left[V(x_1) - V(x_2) \right]_{\parallel}^{3} \right\rangle \sim (x_1 - x_2)$$

If there is no intermittency, then:

$$\left\langle \left[V(x_1) - V(x_2) \right]_{\parallel}^{2} \right\rangle \sim (x_1 - x_2)^{2/3}$$ and $$E_k \sim |V_k|^2 k^2 \sim k^{-5/3}$$

Kolmogorov spectrum   [Kolmogorov 1941]

Kolmogorov energy cascade



$$E_k \sim |V_k|^2 k^2 \sim k^{-5/3}$$

Energy of an eddy of size $\lambda \sim 1/k$ is $E_\lambda \sim V_\lambda^2 \sim |V_k|^2 k^3$;

it is transferred to a smaller-size eddy during time:

$\tau_\lambda \sim \lambda / V_\lambda \sim V_k^{-1} k^{-5/2}$ - "eddy turn-over" time.

The energy flux, $J_\lambda \sim E_\lambda / \tau_\lambda \sim E_k k^{5/3}$, is constant for the Kolmogorov spectrum!

MHD turbulence

Phenomenology:

Energy $E = \int \left(V^2 + B^2 \right) dx$ is conserved, and cascades toward small scales.

- Is energy transfer time $\tau_\lambda \sim \lambda / V_\lambda$? Not necessarily, since dimensional arguments may not work!

Non-dimensional parameter V_λ / V_A can enter the answer.

- Need to investigate interaction of “eddies” in detail!

This is also the main problem in the theory of weak (wave) turbulence.
(waves is plasmas, water, solid states, liquid helium, etc…)
[Kadomtsev, Zakharov, ... 1960’s]

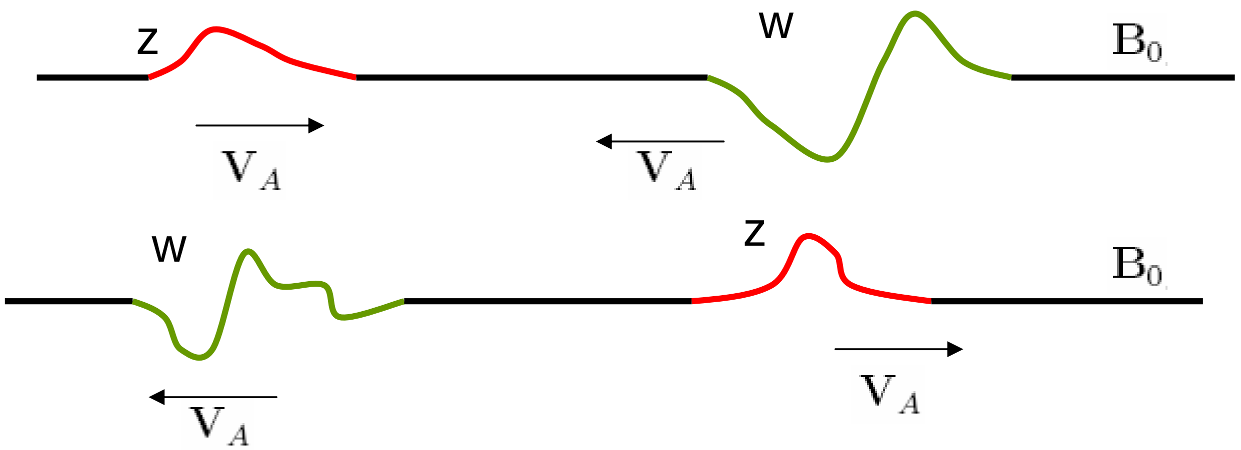
Iroshnikov-Kraichnan spectrum

$$\partial_t \mathbf{z} + (\mathbf{V}_A \cdot \nabla)\mathbf{z} + (\mathbf{w} \cdot \nabla)\mathbf{z} = -\nabla P, \qquad \mathbf{z} = \mathbf{v} - \mathbf{b} \qquad \mathbf{V}_A = \mathbf{B}_0/\sqrt{4\pi\rho}$$

$$\partial_t \mathbf{w} - (\mathbf{V}_A \cdot \nabla)\mathbf{w} + (\mathbf{z} \cdot \nabla)\mathbf{w} = -\nabla P, \qquad \mathbf{w} = \mathbf{v} + \mathbf{b}$$

for $\mathbf{w} = 0$, any function $\mathbf{z} = \mathbf{f}(\mathbf{r} - \mathbf{V}_A t)$ is a solution

for $\mathbf{z} = 0$, any function $\mathbf{w} = \mathbf{g}(\mathbf{r} + \mathbf{V}_A t)$ is a solution



After interaction, shape of each packet changes, but energy does not.

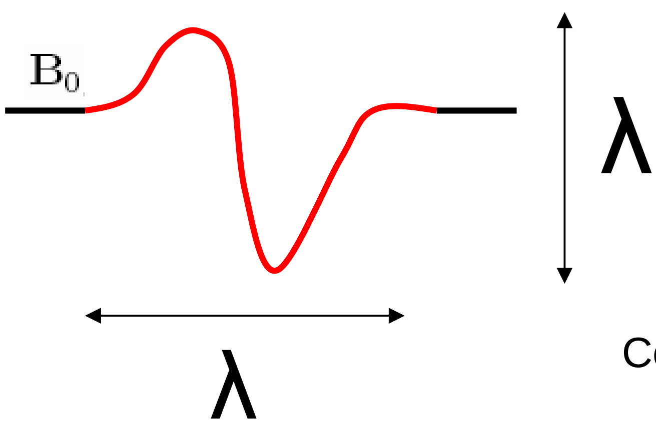
$$\int z^2 d^3x \quad \int w^2 d^3x \quad \Leftrightarrow \quad E = \frac{1}{2}\int (b^2 + v^2) d^3x \quad H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x$$

Iroshnikov-Kraichnan spectrum

$$\partial_t \mathbf{z} + (\mathbf{V}_A \cdot \nabla)\mathbf{z} + (\mathbf{w} \cdot \nabla)\mathbf{z} = -\nabla P,$$
$$\partial_t \mathbf{w} - (\mathbf{V}_A \cdot \nabla)\mathbf{w} + (\mathbf{z} \cdot \nabla)\mathbf{w} = -\nabla P,$$

$$\mathbf{z} = \mathbf{v} - \mathbf{b}$$
$$\mathbf{w} = \mathbf{v} + \mathbf{b}$$

$$\mathbf{V}_A = \mathbf{B}_0/\sqrt{4\pi\rho}$$



typical variations across an eddy:

$$\delta \mathbf{w}_\lambda \sim \delta \mathbf{z}_\lambda \sim \delta v_\lambda \sim \delta b_\lambda$$

during one collision:

$$\Delta \delta v_\lambda \sim (\delta v_\lambda^2/\lambda)(\lambda/V_A)$$

number of collisions required to deform packet considerably:

$$N \sim (\delta v_\lambda/\Delta\delta v_\lambda)^2 \sim (V_A/\delta v_\lambda)^2$$

$$\boxed{\tau_{IK}(\lambda) \sim N\lambda/V_A \sim \lambda/\delta v_\lambda (V_A/\delta v_\lambda)}$$

Constant energy flux: $\delta v_\lambda^2/\tau_{IK}(\lambda) = \text{const}$

$\Rightarrow$ $\delta v_\lambda \propto \lambda^{1/4}$, $\boxed{E_{IK}(k) = |\delta v_k|^2 k^2 \propto k^{-3/2}.}$

[Iroshnikov (1963); Kraichnan (1965)]

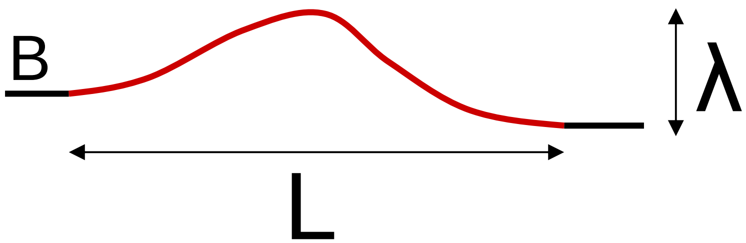
Anisotropy of MHD Turbulence

- Weak incompressible turbulence - analytic derivation (Ng, Bhattacharjee 1997, Galtier et al 2000)
- Weak compressible turbulence - analytic derivation (Chandran 2006)
- Strong compressible turbulence – numerics (Cho, Lazarian 2003)

In weak turbulence, energy of Alfven waves cascades in the field-perpendicular direction

Goldreich-Sridhar theory. Strong Turbulence.

Anisotropy of “eddies”



$$\partial_t \mathbf{z} + (\mathbf{V}_A \cdot \nabla)\mathbf{z} + (\mathbf{w} \cdot \nabla)\mathbf{z} = -\nabla P,$$
$$\partial_t \mathbf{w} - (\mathbf{V}_A \cdot \nabla)\mathbf{w} + (\mathbf{z} \cdot \nabla)\mathbf{w} = -\nabla P,$$

Shear Alfvén waves dominate the cascade: $\delta\mathbf{w}_\lambda \sim \delta\mathbf{z}_\lambda \perp \mathrm{B}$ ⇒ $V_A/l \sim \delta b_\lambda/\lambda$ L>>λ

Critical Balance

$$\tau_{GS}(\lambda) \sim l/V_A \sim \lambda/\delta v_\lambda$$

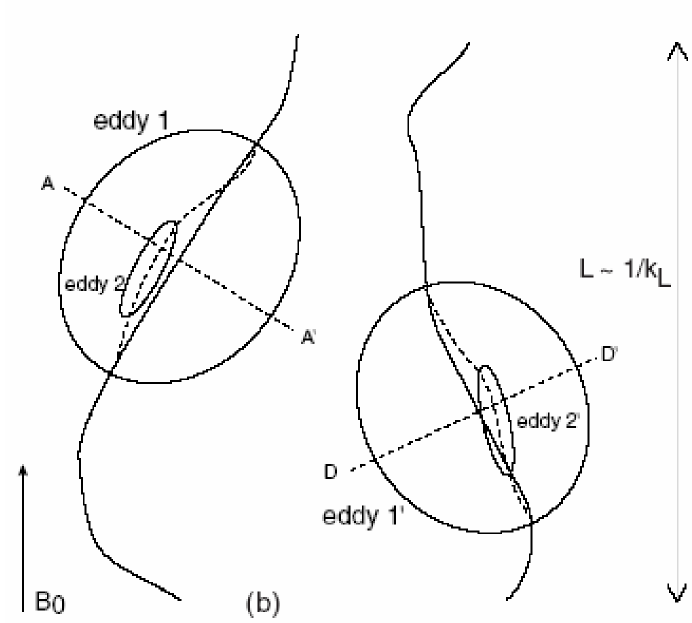
$\delta v_\lambda^2/\tau_{GS}(\lambda) = \mathrm{const}$ ⇒ $\delta v_\lambda \propto \lambda^{1/3}$ ⇒ $E_{GS}(k_\perp) = |\delta v_{k_\perp}|^2 k_\perp \propto k_\perp^{-5/3}$

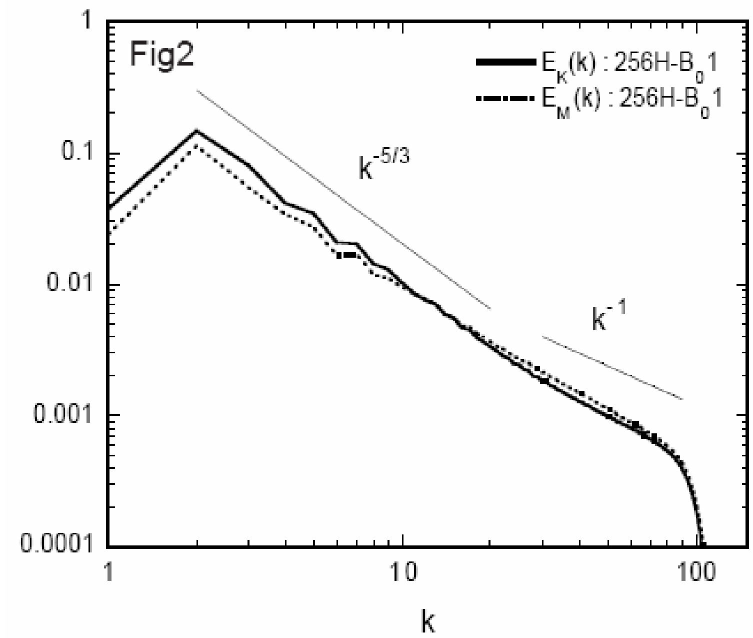
$$l \propto \lambda^{2/3}$$

[Goldreich & Sridhar (1995)]

Goldreich-Sridhar Spectrum in Numerics

Anisotropy of "eddies"

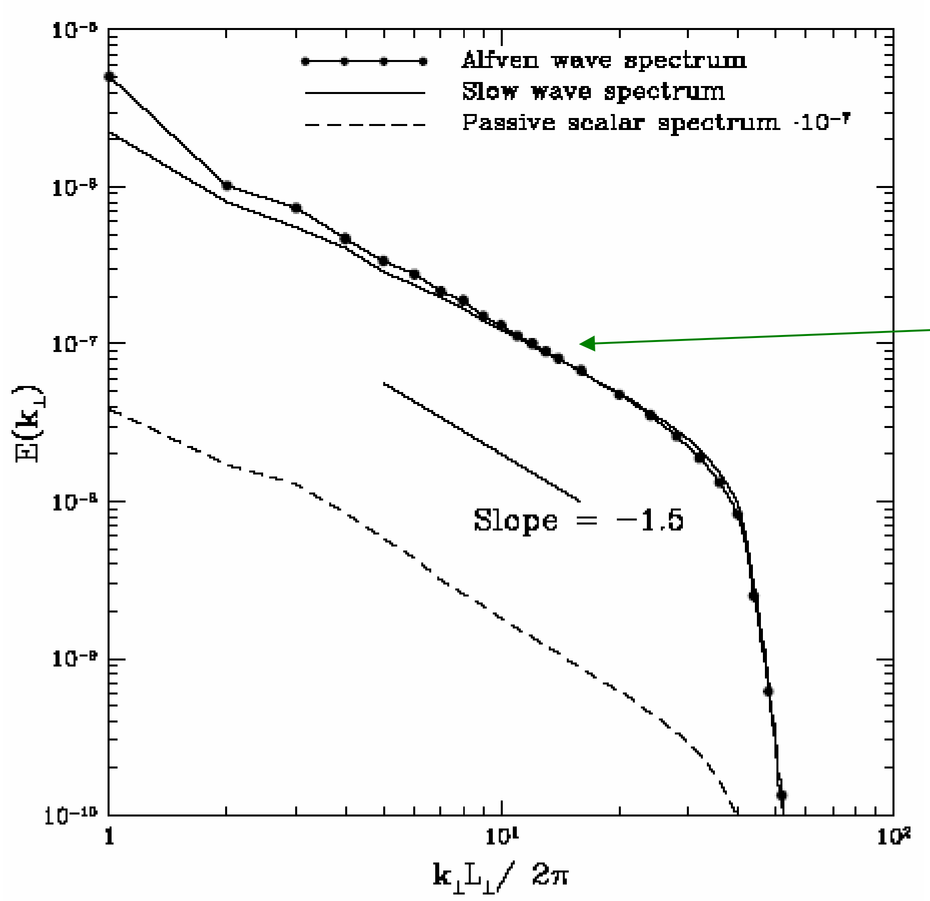




Cho & Vishniac, ApJ, 539, 273, 2000
Cho, Lazarian & Vishniac, ApJ, 564, 291, 2002

Strong Magnetic Field, Numerics

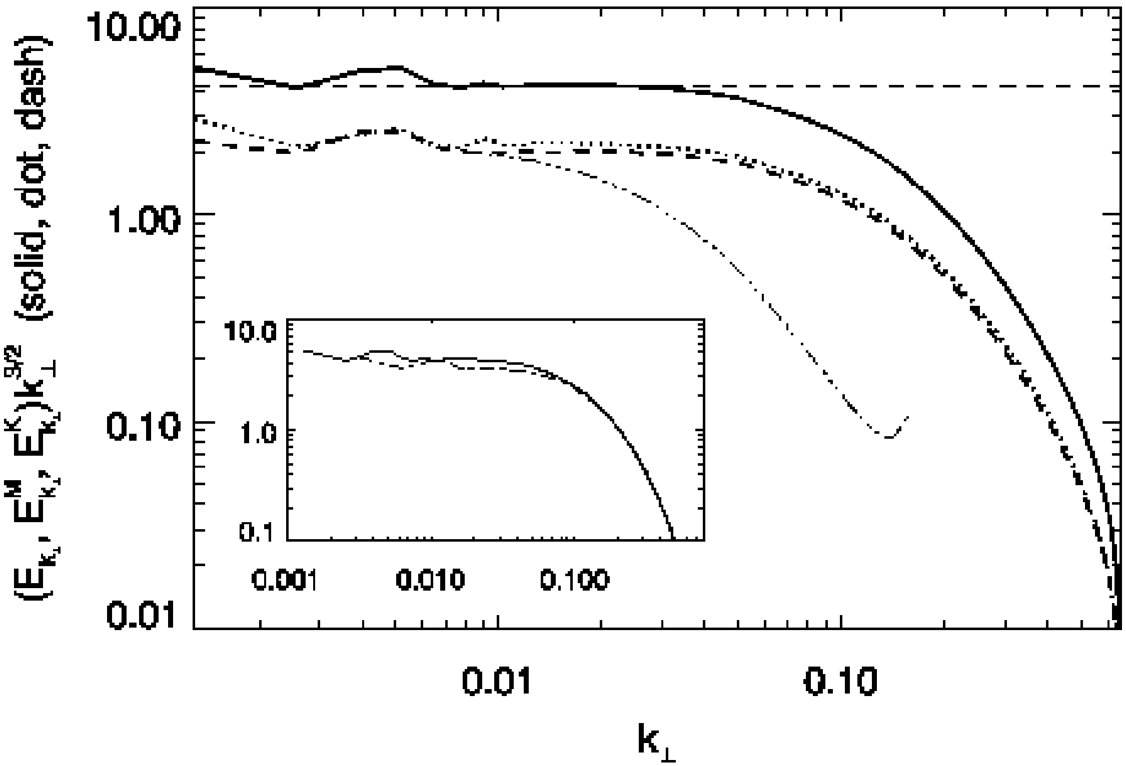
Discrepancies with Goldreich-Sridhar model



Iroshnikov-Kraichnan scaling

[Maron & Goldreich, ApJ 554, 1175, 2001]

Spectrum of magnetized turbulence



Anisotropic, but the scaling is $K^{-3/2}$

Contradicts both GS and IK pictures

Muller & Grappin (2005)

New Model for MHD Turbulence

Analytic Introduction [S.B., ApJ, 626, L37,2005]

Depletion of nonlinear interaction:

$$\partial_t \mathbf{z} + (\mathbf{V}_A \cdot \nabla)\mathbf{z} + (\mathbf{w} \cdot \nabla)\mathbf{z} = -\nabla P,$$
$$\partial_t \mathbf{w} - (\mathbf{V}_A \cdot \nabla)\mathbf{w} + (\mathbf{z} \cdot \nabla)\mathbf{w} = -\nabla P,$$

① ②

$$(\mathbf{w} \cdot \nabla)\mathbf{z} \sim (\mathbf{z} \cdot \nabla)\mathbf{w} \sim (\delta v_\lambda^2/\lambda)(\delta v_\lambda/V_A)^\alpha$$

← Nonlinear interaction is depleted

$$\tau_N(\lambda) \sim (\lambda/\delta v_\lambda)\,(V_A/\delta v_\lambda)^\alpha$$

← Interaction time is increased

For $\delta v_\lambda < V_A$ perturbation cannot propagate along the B-line faster than V , therefore, correlation length along the line is $l \sim V_A \tau_N(\lambda)$

This balances terms ① and ② in the MHD equations, as in the Goldreich-Sridhar picture, however, the geometric meaning is different.

Depletion of Nonlinearity in MHD Turbulence

Analytic Introduction

$$(\mathbf{w}\cdot\nabla)\mathbf{z} \sim (\mathbf{z}\cdot\nabla)\mathbf{w} \sim (\delta v_\lambda^2/\lambda)(\delta v_\lambda/V_A)^\alpha$$ ← Nonlinear interaction is depleted

$$\tau_N(\lambda) \sim (\lambda/\delta v_\lambda)\,(V_A/\delta v_\lambda)^\alpha$$ ← Interaction time is increased

Constant energy flux, $\delta v_\lambda^2/\tau_N(\lambda) = \text{const}$ ⇒

Goldreich-Sridhar scaling corresponds to α=0:

$$\tau_N \sim \lambda^{2/3} \quad l \sim \lambda^{2/3} \quad \delta v_\lambda \propto \lambda^{1/3} \quad \delta v_l \propto l^{1/2}$$

"Iroshnikov-Kraichnan" scaling is reproduced for α=1:

$$\tau_N \sim \lambda^{1/2} \quad l \sim \lambda^{1/2} \quad \delta v_\lambda \propto \lambda^{1/4}, \quad \delta v_l \propto l^{1/2}$$

Explains numerically observed scalings for strong B-field !

[Maron & Goldreich, ApJ 554, 1175, 2001]

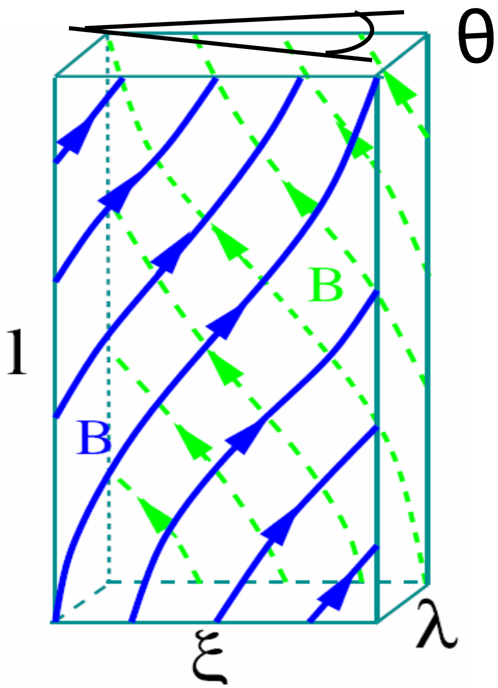
[Müller, Biskamp, Grappin PRE, 67, 066302, 2003]

Dynamic Alignment in MHD Turbulence

Depletion of nonlinearity

$\delta v_\lambda \propto \lambda^{1/4}$

line displacement: $\xi \propto \delta b_\lambda l$



$$\partial_t \mathbf{z} + (\mathbf{V}_A \cdot \nabla)\mathbf{z} + (\mathbf{w} \cdot \nabla)\mathbf{z} = -\nabla P,$$
$$\partial_t \mathbf{w} - (\mathbf{V}_A \cdot \nabla)\mathbf{w} + (\mathbf{z} \cdot \nabla)\mathbf{w} = -\nabla P,$$

In our "eddy", **v** and **b** are aligned within small angle θ_λ. One can check that:

$$(\mathbf{w} \cdot \nabla)\mathbf{z} \sim \delta v_\lambda^2 \theta_\lambda / \lambda$$

Nonlinear interaction is reduced!

In our theory, this reduction of interaction is:

$$\theta_\lambda \sim \lambda/\xi \sim \lambda^{1/4}$$

Alignment is scale-dependent

Energy spectrum is $E(k_\perp) \propto k_\perp^{-3/2}$

Scale-Dependent Dynamic Alignment in MHD Turbulence

Geometric Meaning

Goldreich-Sridhar 1995 “eddy”:

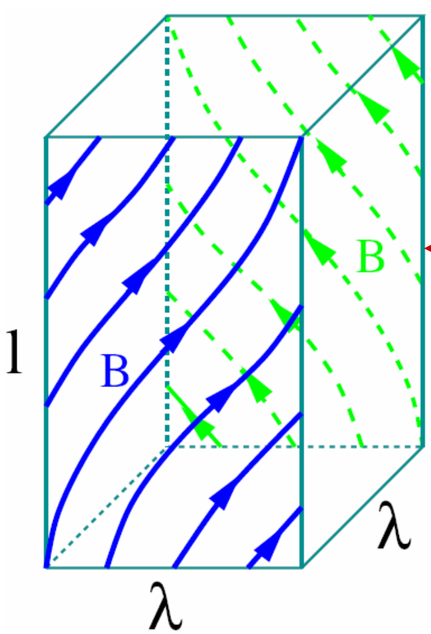
$\delta v_\lambda \propto \lambda^{1/3}$ $\quad l \sim \lambda^{2/3}$

line displacement: $\xi \propto \delta b_\lambda l \sim \lambda$

Our “eddy”:

$\delta v_\lambda \propto \lambda^{1/4}$ $\quad l \sim \lambda^{1/2}$

line displacement: $\xi \propto \delta b_\lambda l \sim \lambda^{3/4}$

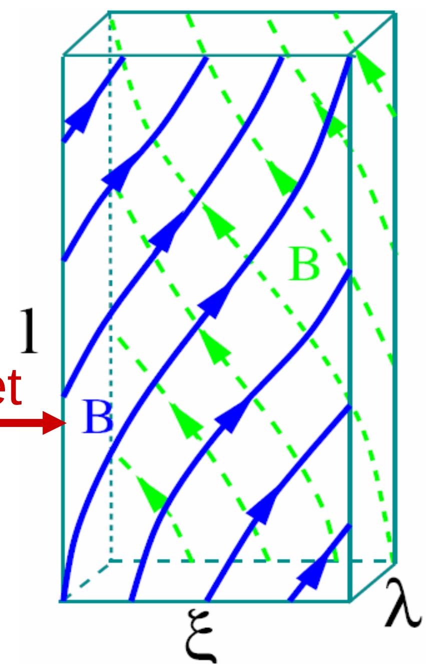


As the scale decreases, λ→0,

turns into filament

turns into current sheet

agrees with numerics!



Physical meaning of dynamic alignment

Two conserved integrals

$$E = \frac{1}{2}\int (b^2 + v^2) d^3x$$ energy

$$H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x$$ cross helicity

Both quantities cascade toward small scales.
Requirement of **two** cascades leads to two relations:

$$\theta_\lambda \propto \lambda^{1/4}$$
$$\delta v_\lambda \propto \lambda^{1/4}$$

No intermittency is required in this model!

How to measure dynamic alignment?

Structure functions

$$\delta \mathbf{b}_r = \mathbf{b}(\mathbf{x}+\mathbf{r}) - \mathbf{b}(\mathbf{x}) \qquad \delta \mathbf{v}_r = \mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})$$

$$\delta \tilde{\mathbf{b}}_r = \delta \mathbf{b}_r - (\delta \mathbf{b}_r \cdot \mathbf{n})\mathbf{n} \qquad \delta \tilde{\mathbf{v}}_r = \delta \mathbf{v}_r - (\delta \mathbf{v}_r \cdot \mathbf{n})\mathbf{n}$$

$$\mathbf{n} = \mathbf{B}(x)/|\mathbf{B}(x)|$$

$$S_{cross}(r) = \langle |\delta \tilde{\mathbf{v}}_r \times \delta \tilde{\mathbf{b}}_r| \rangle$$

$$S_2(r) = \langle |\delta \tilde{\mathbf{v}}_r||\delta \tilde{\mathbf{b}}_r| \rangle$$

Alignment angle: $\theta_r \approx \sin(\theta_r) \equiv S_{cross}(r)/S_2(r)$

should scale as θ_r / $r^{1/4}$

J. Mason, F. Cattaneo & S. B. (2006)

Joanne Mason, next talk
Jean Carlos Perez, poster session

How to measure dynamic alignment?

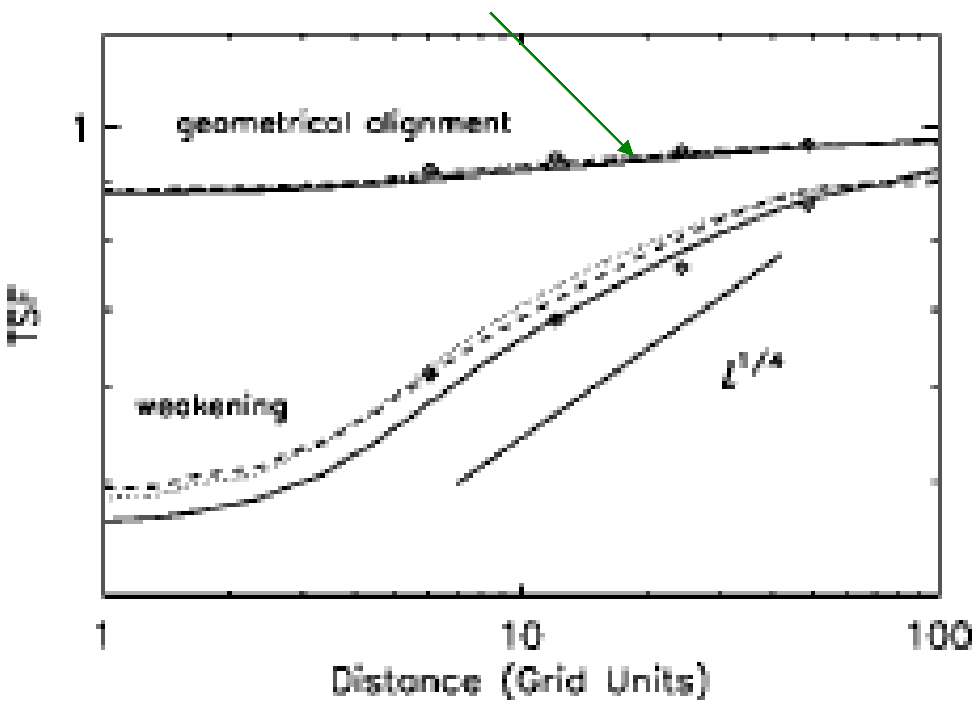
δv_λ / $\lambda^{1/4}$ δb_λ / $\lambda^{1/4}$

typical fluctuations at scale λ, say rms values.

The geometric alignment should be measured between such **dynamically relevant** fluctuations

For example, it would be incorrect to measure the geometric alignment as:

$\sin(\theta_\lambda)$= hn_h£n_vi n_h ´ $\delta b_\lambda/|\delta b_\lambda|$ n_v´ $\delta v_\lambda/|\delta v_\lambda|$



Beresnyak & Lazarian ApJ 640 (2006) L175:

"*geometric alignment cannot describe weakening of interaction,*" rather… "*polarization intermittency*"

Yes, it can, if we measure it correctly!

see Joanne's talk

MHD spectrum. Observations.

Spectrum of electron density fluctuations inferred from observations is broadly consistent with -5/3, however, there are notable exceptions:

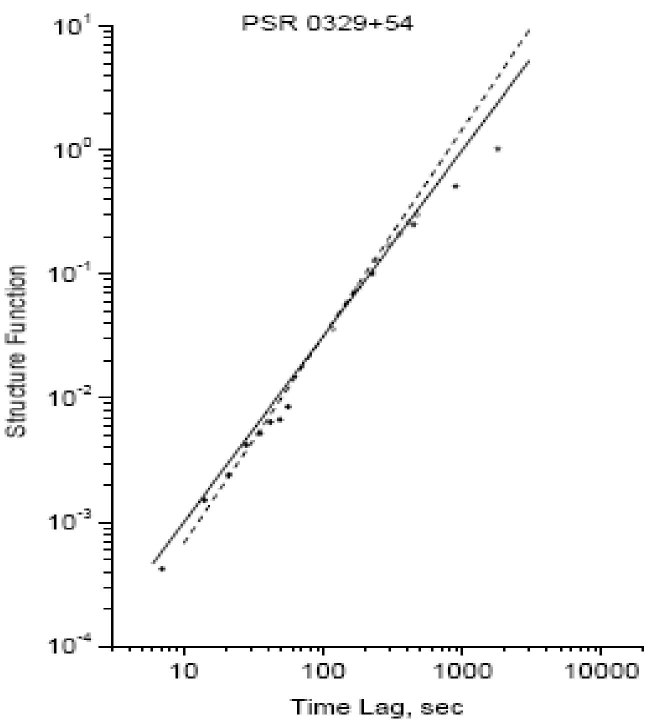


Fig. 8. Time structure function of phase fluctuations at $f_0 = 1$ GHz as compiled from the following data points 1.) 102 MHz observations: filled circles; 2.) 610 MHz: open circles; 3.) 5 GHz: stars. The solid line corresponds to the best fit to the first points of the structure functions. The slope is 1.50. The dashed line corresponds to $n = 1.67$ (Kolmogorov model).

Shishov et al, A&A 404 (2003) 557

MHD spectrum. Observations.

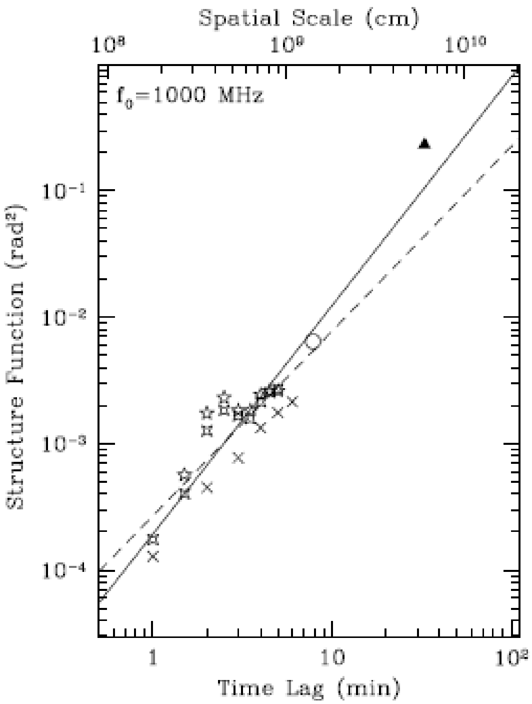


Fig. 2. The time structure function of phase fluctuations for PSR J0437−4715 reduced to the reference frequency f_0 = 1000 MHz, as compiled from the observations. Open symbols are as in Figure 1. The solid triangle indicates scintillation of the quasar PKS 0405−385 Rickett et al. (2002). The dashed line indicates the best fit to data for the pulsar (index $\alpha_1 = 1.46 \pm 0.20$); the solid line corresponds to the best-fitting power-law to all points (index $\alpha_2 = 1.8 \pm 0.15$). Top x-axis corresponds to spatial scale of the inhomogeneities.

Smirnova, Gwinn, Shishov
astro-ph/0603490

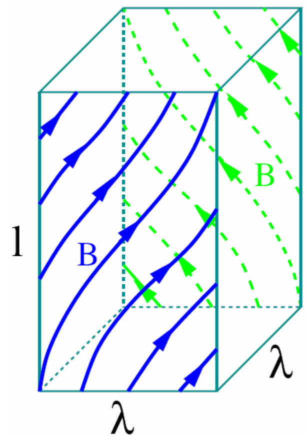
Summary and Discussions

1. Weak large-scale field: $B << \rho V^2$

$\delta v_\lambda \propto \lambda^{1/3}$ $l \sim \lambda^{2/3}$ $\xi \propto \delta b_\lambda l \sim \lambda$

dissipative structures: filaments

energy spectrum: $E(K) \sim K_\perp^{-5/3}$



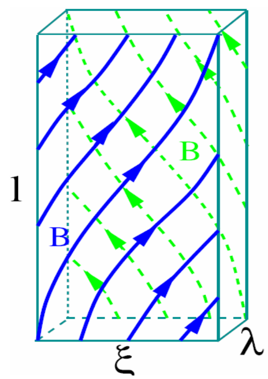
[Goldreich & Sridhar' 95]

2. Strong large-scale field: $B >> \rho V^2$

$\delta v_\lambda \propto \lambda^{1/4}$ $l \sim \lambda^{1/2}$ $\xi \propto \delta b_\lambda l \sim \lambda^{3/4}$

dissipative structures: current sheets

energy spectrum: $E(K) \sim K_\perp^{-3/2}$



$\theta_\lambda \sim \lambda/\xi \sim \lambda^{1/4}$

scale-dependent dynamic alignment

3. The spectrum of MHD turbulence may be $E \sim K_\perp^{-3/2}$, but in case **1,** resolution of numerical simulations is not large enough to observe it...