Weakly Compressible MHD Turbulence in the Solar Wind and the Reversed Field Pinch

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Weakly Compressible MHD Equations

As in a reversed-field pinch, turbulent plasmas in the solar wind contain:

- a significant directed magnetic field with background spatial homogeneities
- plasma $\beta \leq 1$

Effects on turbulence

- Weakly compressible
- Anisotropic, with slow spatial variation parallel and rapid variation perpendicular to field lines

Outline

- Four-field equations to describe weakly compressible MHD turbulence using the Mach number of the turbulence as an asymptotic expansion parameter. Despite the apparent difference in expansion procedure, it turns out that these equations are identical to the equations for RFP pressure-driven turbulence
- Relation of these equations to the Nearly Incompressible MHD (NI-MHD) equations of G. Zank and W. Matthaeus.

Two applications

- Density fluctuations in the solar wind as measured by Helios.
- Variance anisotropy, as measured by ACE, and calculated using the invariance principle approach (Connor and Taylor, 1984; Bhattacharjee and Hameiri, 1987)

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Start with compressible MHD (in the plasma frame)

$$\rho\left(\frac{\partial}{\partial t}+\mathrm{v}\cdot\nabla\right)\mathrm{v}=-\nabla p+\frac{1}{4\pi}(\nabla\times\mathrm{B})\times\mathrm{B}$$

$$\frac{\partial\mathrm{B}}{\partial t}+\nabla\times(\mathrm{B}\times\mathrm{v})=0$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathrm{v})=0$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right)=0$$

Scaling every dependent variable by its characteristic value, these equations can be cast in dimensionless form.

Define:

$$V_A^2\equiv B_0^2\,/\,4\pi\rho_0$$

$$M^2\equiv v_0^2\,/\,c_s^2$$

$$M_A^2\equiv v_0^2\,/\,V_A^2$$

$$\beta\equiv 4\pi p_0\,/\,B_0^2=M_A^2\,/\,\gamma M^2$$

Zank-Matthaeus ordering

$$\rho\left(\frac{\partial}{\partial t}+\mathrm{v}\cdot\nabla\right)\mathrm{v}=\frac{1}{\varepsilon^2}\left(-\nabla p+\frac{1}{\beta}(\nabla\times\mathrm{B})\times\mathrm{B}\right)$$

where $\varepsilon\equiv\sqrt{\gamma}M$ is a small parameter for low Mach number

Expand

$$\begin{aligned}\mathrm{B}&=\mathrm{B}_0+\mathrm{B}_1+\ldots\\ \mathrm{v}&=\qquad\ \mathrm{v}_1+\ldots\\ \rho&=\rho_0+\rho_1+\ldots\\ p&=p_0+p_1+\ldots\end{aligned}$$

To leading order

$$\nabla p_0=\frac{1}{\beta}(\nabla\times\mathrm{B}_0)\times\mathrm{B}_0$$

ZM assume that

$$p_0=1,$$

$$\mathrm{B}_0=\hat{z}$$

i. e. uniform background magnetic field

which is a special case

Higher order

$\mathrm{v} \sim O(\varepsilon)$, $\nabla_\perp \sim O(1/\varepsilon)$ and $\nabla_\parallel \sim O(1)$

To $O(\varepsilon^2)$

$$p_1 + \mathrm{B}_0 \cdot \mathrm{B}_1 = 0$$

From equation of state, to $O(1)$

$$\nabla_\perp \cdot \mathrm{v}_1 = 0 \text{ --- weakly compressible}$$

$$\nabla_\perp \cdot \mathrm{B}_1 = 0$$

Therefore,

$$\mathrm{B}_1 = \nabla_\perp A \times \mathrm{b} - p_1 \mathrm{b}$$

$$\mathrm{v}_1 = \nabla_\perp \varphi \times \mathrm{b} - v_1 \mathrm{b}$$

where $\mathrm{b} \equiv \mathrm{B}_0 / B_0^2$

Four field variables:

A : flux function, p_1 : parallel magnetic field,

φ : stream function, v_1 : parallel flow

Four-field system for weakly compressible MHD

$$\frac{dA}{dt} = \mathrm{B}_0 \cdot \nabla \varphi + \eta \nabla_\perp^2 A$$

$$\frac{dp}{dt} = -\mathrm{v} \cdot \nabla p_0 + \bar{\gamma}[2\mathrm{v} \cdot \nabla P + Dv + \eta \nabla_\perp^2 p]$$

$$\rho_0 \frac{d\Omega}{dt} = DJ + 2\mathrm{b} \times \nabla P \cdot \nabla_\perp p - (\mathrm{b} \cdot \nabla B_0^2) J + \nu \nabla_\perp^2 \Omega$$

$$\rho_0 \frac{dv}{dt} = Dp + \mathrm{B}_1 \cdot \nabla p_0 + \nu \nabla_\perp^2 v$$

with

$$\mathrm{B}_1 = \nabla_\perp A \times \mathrm{b} - p\mathrm{b},\ \mathrm{v} = \nabla_\perp \varphi \times \mathrm{b} - v\mathrm{b}$$

$$\Omega \equiv -\nabla_\perp^2 \varphi,\ J \equiv -\nabla_\perp^2 A,$$

$$P \equiv p_0 + B_0^2 / 2,\ \bar{\gamma} \equiv \gamma p_0 / (\gamma p_0 + B_0^2),$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathrm{v} \cdot \nabla_\perp,\ D \equiv (\mathrm{B}_0 + \mathrm{B}_1) \cdot \nabla,$$

$$\nabla_\perp \equiv \nabla - \hat{\mathrm{B}}_0 \hat{\mathrm{B}}_0 \cdot \nabla,\ \hat{\mathrm{B}}_0 \equiv \mathrm{B}_0 / B_0,$$

$$\rho_0 = p_0^{1/\gamma}.$$

Note that the background field satisfies equilibrium conditic

$$\nabla p_0 = (\nabla \times \mathrm{B}_0) \times \mathrm{B}_0$$

so that B_0 is not uniform in general.

Remarks

- Let

  $$B = \frac{\hat{z}}{\sqrt{\beta}} + B_s(x)$$

  If $B_s(x) \rightarrow 0$, the four-field system reduces to RMHD with two field variables A and φ.

- The case with $\beta << 1$ is effectively the same as $B_s(x) \rightarrow 0$, so we also recover RMHD.

- If $\beta \sim 1$ and $B_s(x) \neq 0$, compressibility enters at leading order, and the equations differ from those of Zank and Matthaeus who claim that "compressible effects ride parasitically on the back of the 2D incompressible flow field."

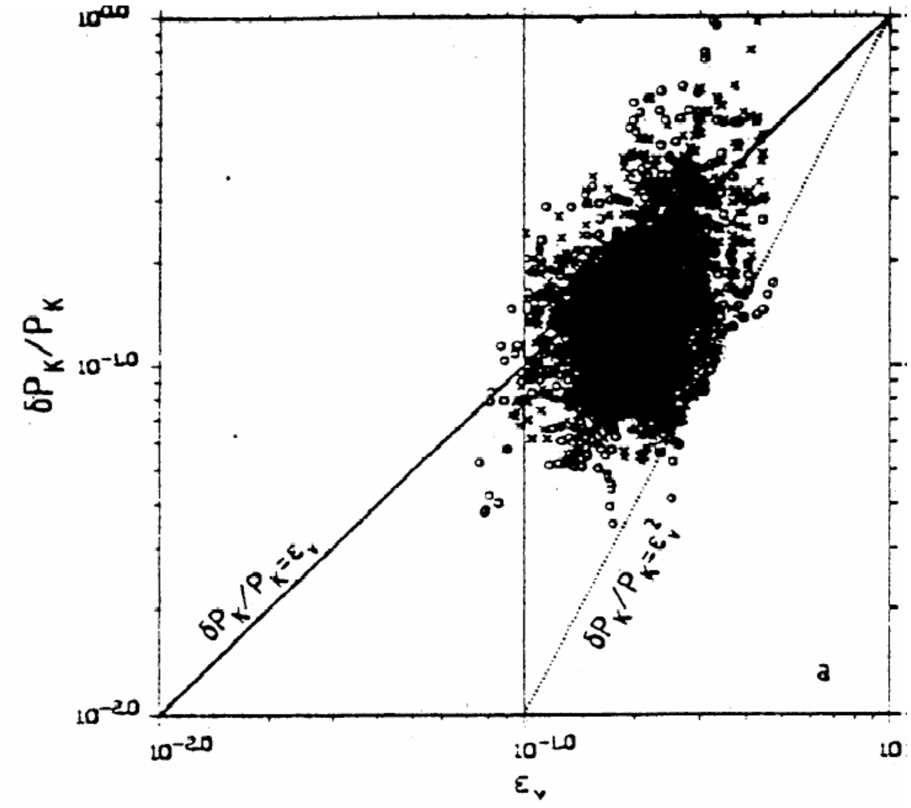


Figure 6. The normalized pressure fluctuations from Helios 1 (indicated by circles) and Helios 2 (indicated by crosses) as a function of the effective Mach number, reproduced from [*uT d sM* ,1994].

Helios I observations

(Tu and Marsch, 1994):

Pressure or density fluctuations can be much larger than are predicted by NI-MHD theory.

Two dimensional equations

Consider

$$\mathrm{B}_0 = \frac{1}{\sqrt{\beta}}\hat{\mathrm{z}} + \nabla_{\perp 0} A_0(x,y) \times \hat{\mathrm{z}}$$

with

$$\nabla_{\perp 0} = \hat{\mathrm{x}}\frac{\partial}{\partial x} + \hat{\mathrm{y}}\frac{\partial}{\partial y}, \quad \nabla^2_{\perp 0} A_0 = -k^2 A_0$$

so that

$$p_0 = 1 + k^2 A_0^2 / 2$$

2D runs with 256×256 resolution

$$A_0 = a[\cos(2\pi x) - \cos(2\pi y)]$$

$$\beta = 1$$

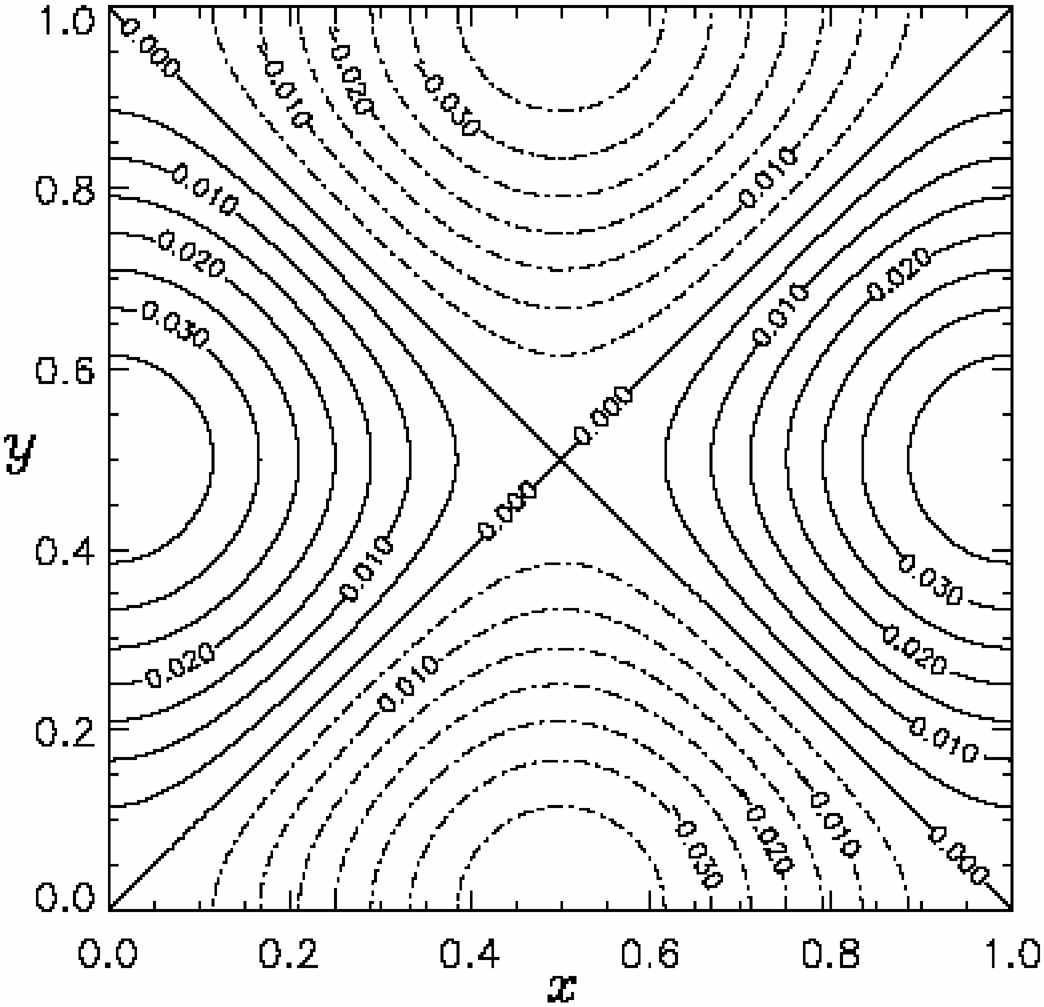


FIG. 1.—Contour plot of the background field A_0 (given by eq. [28]) with $a = 0.04$.

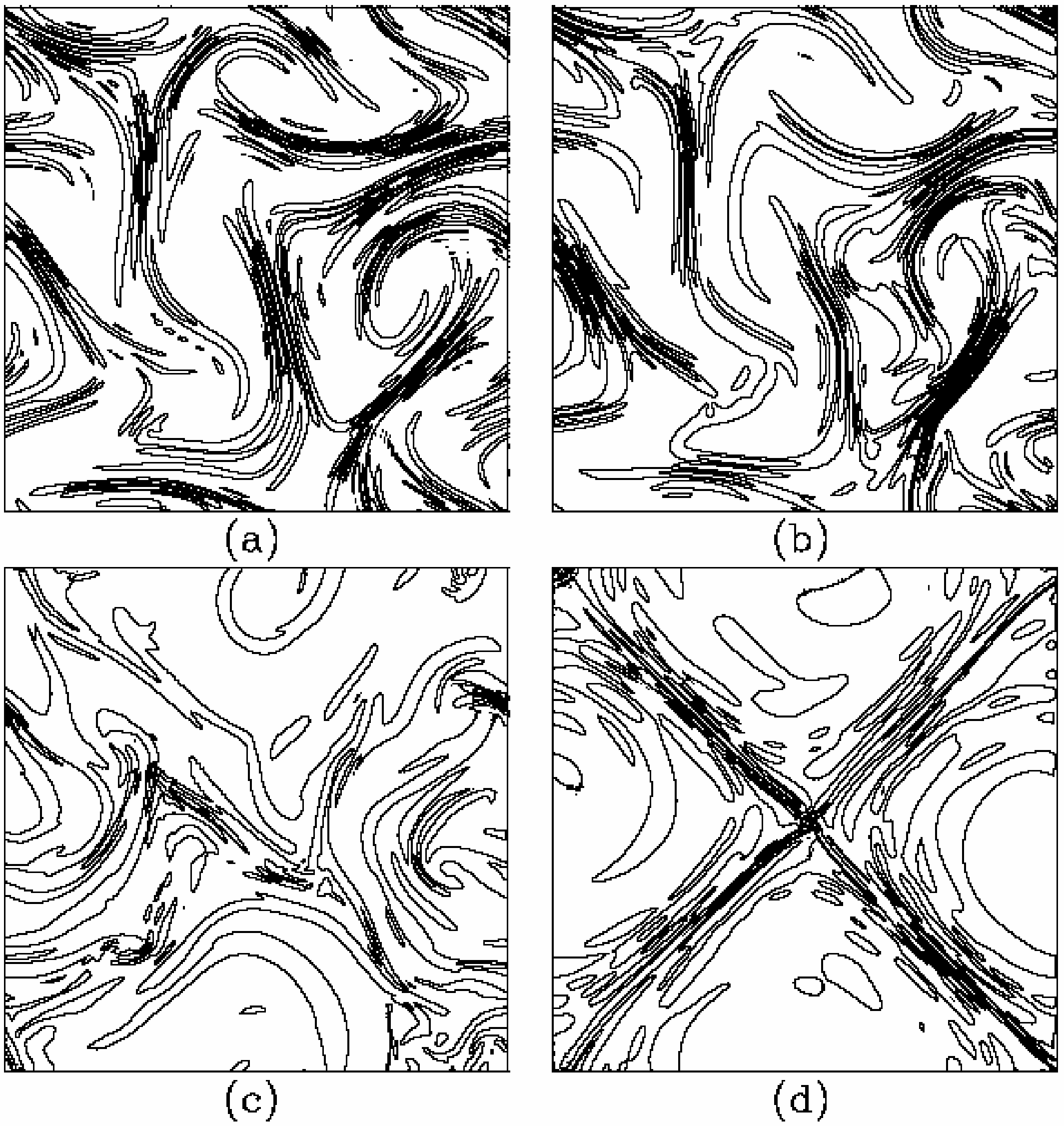


FIG. 5.—Contour plots of current density for different values of *a* at the time *t* near the instant when $J_{\max}$ attains its highest value (see Fig. 4). (*a*) $a = 10^{-6}$, (*b*) $a = 10^{-4}$, (*c*) $a = 10^{-3}$, (*d*) $a = 10^{-2}$.

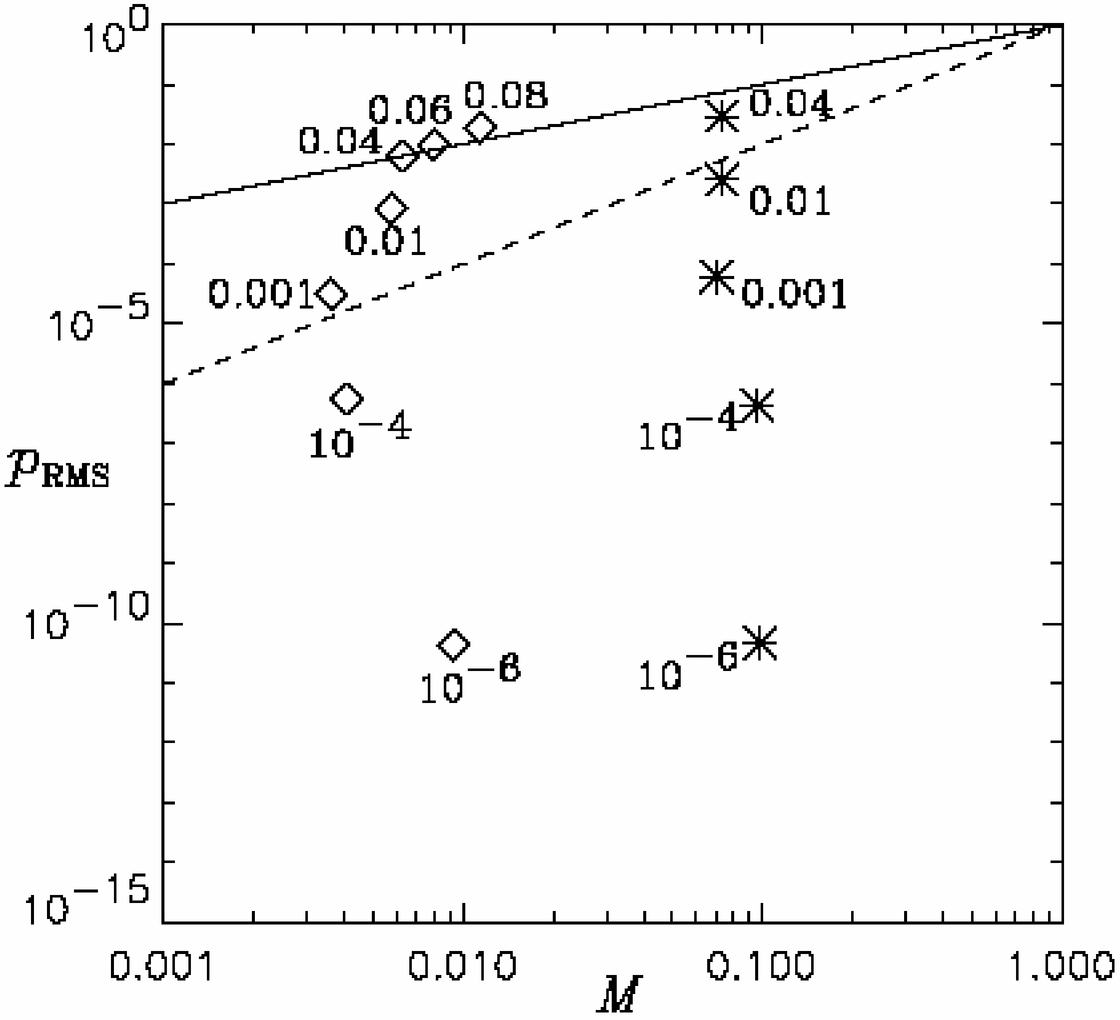


FIG. 6.—Level of the rms value of the first-order pressure p_{rms} in the quasi-saturated state as a function of the Mach number M for different values of a, for $M(t = 0) \approx 0.01$ (*diamonds*) and $M(t = 0) \approx 0.1$ (*stars*). Solid line corresponds to $p_{\mathrm{rms}} = M$, and dashed line to $p_{\mathrm{rms}} = M^2$.

Fluctuation level as a function of spatial inhomogeneity parameter δ : comparison with Helios observations

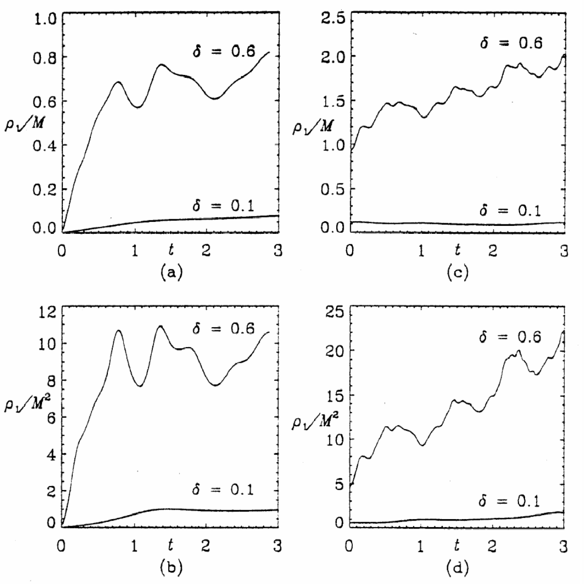


Figure 5. Average density fluctuations divided by Mach number *M* and its square as functions of time from simulations of the four-field equations [frames (a) and (b)] and the compressible MHD equations [frames (c) and (d)].

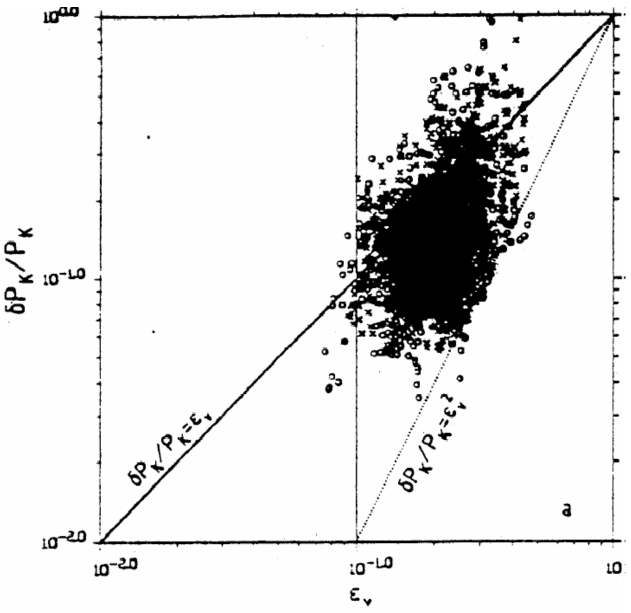


Figure 6. The normalized pressure fluctuations from Helios 1 (indicated by circles) and Helios 2 (indicated by crosses) as a function of the effective Mach number, reproduced from [*T d M* ,1994].

Smith, Vasquez, and Hamilton 2006

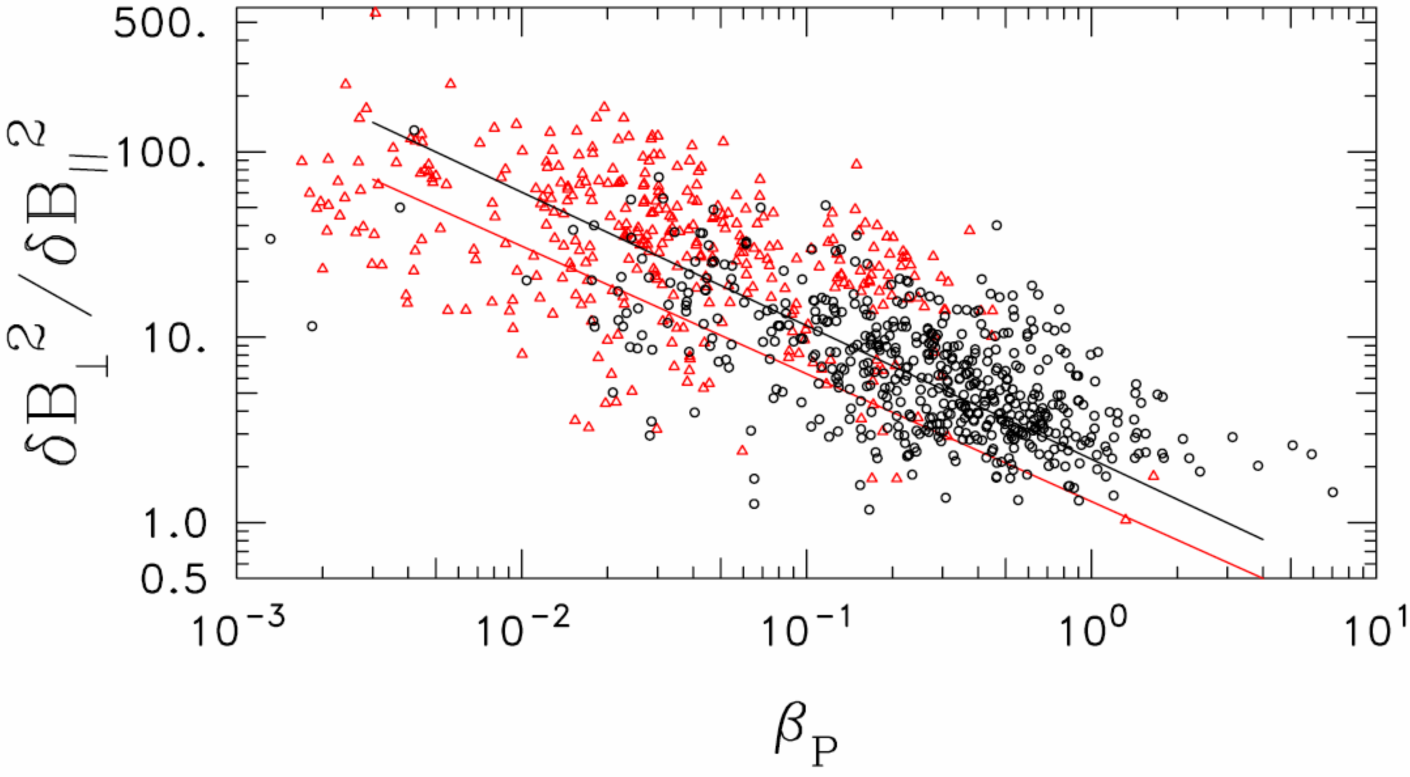


Figure 3. Scatter plot of computed anisotropy vs. β_P. Red denotes cloud observations and black represents open field lines. The fit line parameters are given in Table 1.

Table 1. Figure 3 fit parameters.

Population	Fit Function	χ_r^2
open	$\delta B_\perp^2/\delta B_\parallel^2 = (2.12 \pm 0.1)\beta^{-0.72\pm0.04}$	1.07
clouds IMF	$\delta B_\perp^2/\delta B_\parallel^2 = (1.29 \pm 0.2)\beta^{-0.69\pm0.08}$	2.59

Model of local pressure-driven turbulence based on the weakly compressible MHD equations

- Solar wind turbulence is a bath of spaghetti-like cylindrical flux tubes, containing plasma with inhomogeneous magnetic and pressure profiles that are unstable with respect to pressure-driven instabilities (such as interchange/ballooning).
- The weakly compressible MHD equations can then be shown to depend on local parameters such as plasma $\beta,\ \kappa = (r/p)(dp/dr)$, driven by local imhomogeneities.
- The properties of the turbulent fluctuations can be determined from the Invariance Principle.

Invariance Principle

"If the equations that describe turbulence in the solar wind are invariant under a local scale transformation, then any fluctuation spectra calculated from them must exhibit the same invariance."

(Lamb 1932, Landau and Lifshitz 1959, Sedov 1959, Connor and Taylor 1977, Bhattacharjee and Hameiri 1988)

See Sedov, [illegible] [illegible]
[illegible] 1959

$$\frac{d\tilde{A}}{d\tau} = \frac{\partial\tilde{\phi}}{\partial\theta} + \frac{1}{S}\Delta_\perp \tilde{A}\,, \tag{15}$$

$$\frac{d}{d\tau}\Delta_\perp \tilde{\phi} = \frac{\partial}{\partial\theta}\Delta_\perp \tilde{A} + \{\tilde{A},\Delta_\perp \tilde{A}\} - \frac{\partial\tilde{p}}{\partial y}\,, \tag{16}$$

$$\begin{aligned}\frac{d\tilde{p}}{d\tau} = \frac{\gamma p}{\gamma p + B^2}\Bigg(\Sigma\frac{\partial\tilde{\phi}}{\partial y} + \frac{\partial\tilde{v}}{\partial\theta}\\ + \{\tilde{A},\tilde{v}\} + \frac{1}{S}\Delta_\perp \tilde{p}\Bigg) - K\frac{\partial\tilde{\phi}}{\partial y}\,,\end{aligned} \tag{17}$$

$$\frac{d\tilde{v}}{d\tau} = \frac{\partial\tilde{p}}{\partial\theta} + \{\tilde{A},\tilde{p}\} + K\frac{\partial\tilde{A}}{\partial y}\,, \tag{18}$$

where we have used the following definitions:

$$\begin{aligned}\{f,g\} &= \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\,,\\ \frac{df}{d\tau} &= \frac{\partial f}{\partial\tau} + \{\tilde{\phi},f\}\,,\\ \Delta_\perp &= \left(\frac{\partial}{\partial x} - \theta\frac{\partial}{\partial y}\right)^2 + \frac{1}{\sigma^2}\frac{\partial^2}{\partial y^2}\,.\end{aligned} \tag{19}$$

The large parameter S, the magnetic Reynolds number, may be eliminated from the system by using the following transformation:

$$\begin{aligned}&x\to x/\sqrt{S}\,,\quad y\to y/\sqrt{S}\,,\quad \tilde{A}\to\tilde{A}/S,\\ &\tilde{\phi}\to\tilde{\phi}/S,\quad \tilde{p}\to\tilde{p}/\sqrt{S}\,,\quad \tilde{v}\to\tilde{v}/\sqrt{S}\,.\end{aligned} \tag{20}$$

Here

$$
\begin{aligned}
&B=|\mathbf{B}|,\quad \mu=rB_z/B_\theta,\quad \sigma=(B_\theta/B)\mu'\,,\\
&K=-2rp'/(B_\theta^2\,\mu'^2),\quad \Sigma=4/\mu'^2,\quad S=rB_\theta/\eta\sqrt{\rho}\,,
\end{aligned}
\tag{12}
$$

Pressure-driven Turbulence

$$\Delta_\perp \tilde{A}_0 = 0,\quad \frac{1}{S}\Delta_\perp \tilde{A}_1 = \frac{d\tilde{A}_0}{d\tau} - \frac{\partial\tilde{\phi}}{\partial\theta}\,.$$

where $\tilde{A} = \tilde{A}_0 + \tilde{A}_1$

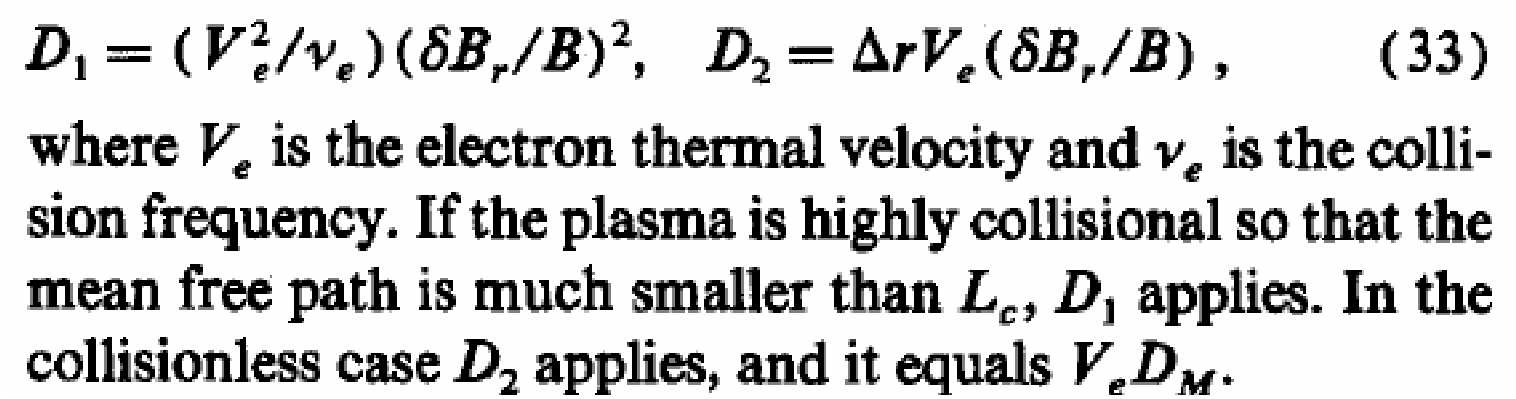
Two branches:

Electrostatic $\tilde{A}_0$ = 0

Electromagnetic $\tilde{A}_0 \neq 0.$

Earlier results on RFP pressure-driven turbulence

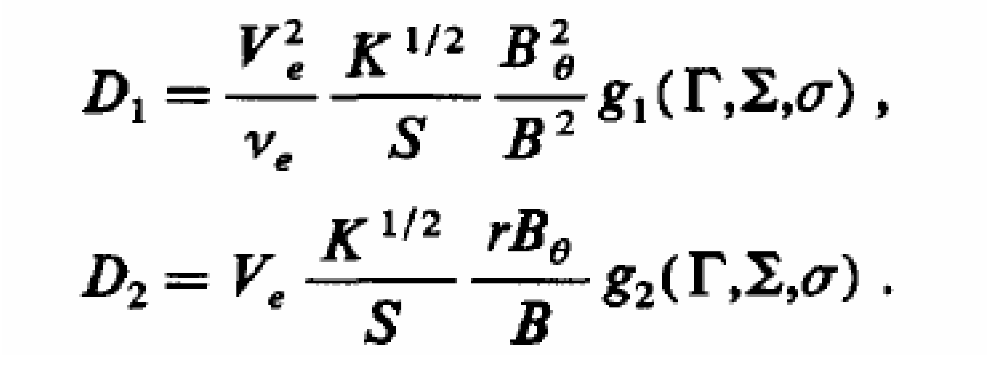
(Bhattacharjee and Hameiri, 1988)



$$D_1 = (V_e^2/\nu_e)(\delta B_r/B)^2, \quad D_2 = \Delta r V_e (\delta B_r/B)\ , \qquad (33)$$

where V_e is the electron thermal velocity and ν_e is the collision frequency. If the plasma is highly collisional so that the mean free path is much smaller than L_c, D_1 applies. In the collisionless case D_2 applies, and it equals $V_e D_M$.

Invariance Principle yields



$$D_1 = \frac{V_e^2}{\nu_e} \frac{K^{1/2}}{S} \frac{B_\theta^2}{B^2} g_1(\Gamma,\Sigma,\sigma)\ ,$$

$$D_2 = V_e \frac{K^{1/2}}{S} \frac{r B_\theta}{B} g_2(\Gamma,\Sigma,\sigma)\ .$$

Scaling predictions of weakly compressible MHD theory

- Electrostatic pressure-driven modes

$$\delta B_\perp \sim \beta^{7/4},\ \delta B_{||} \sim \beta^{7/4}$$

Variance anisotropy $\delta B_\perp^2 / \delta B_{||}^2 \sim 1$

- Electromagnetic pressure-driven modes

$$\delta B_\perp \sim \beta^{3/4},\ \delta B_{||} \sim \beta^{7/4}$$

Variance anisotropy $\delta B_\perp^2 / \delta B_{||}^2 \sim 1/\beta^2$

Theory predicts variance anisotropy $\sim 1/\beta^\nu,\ 0 < \nu < 2$

Observations of Smith et al. $\sim 1/\beta^\nu,\ \nu \sim 0.7$