

# Dynamo action in low-Prandtl helical turbulence

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## Introduction and Outline

- Turbulent dynamo converts kinetic energy of a turbulent plasma flow into magnetic energy by random stretching of magnetic field lines frozen into the plasma. Dynamo generates and/or redistributes fields in cosmic plasmas. Observed magnetic fields are often correlated at scales larger than the scales of plasma velocities. Such fields can be explained if there is nonzero kinetic helicity,  $H = \int \mathbf{v} \cdot (\nabla \times \mathbf{v}) d^3x \neq 0$ .
- Find a numerical solution for the Kazantsev-Kraichnan model of the kinematic turbulent dynamo with kinetic helicity.
- Focus on low-Prandtl dynamo,  $Pr = \nu/\eta \ll 1$ , hard to simulate.
- **Our results** indicate that, in the case  $Pr = Rm/Re \ll 1$ ,
  1. There is a limited applicability of the conventional alpha-model of a large-scale dynamo action,  $\partial_t \bar{\mathbf{B}} = \alpha \nabla \times \bar{\mathbf{B}} + \beta \nabla^2 \bar{\mathbf{B}}$ .
  2. Field growth rate is  $\propto Rm^{1/2}$  in the limit  $1 \ll Rm \ll Re$ .
  3. Critical  $Rm$  (when dynamo is exited) is much lower in the case when low-Prandtl turbulence has nonzero kinetic helicity.

## Kinematic Kazantsev-Kraichnan model

Consider induction equation

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$

Homogeneous isotropic turbulent velocities are prescribed as being a Gaussian white noise:

$$\begin{aligned} \langle v^i(\mathbf{x}, t) v^j(\mathbf{x}', t') \rangle &= \kappa^{ij}(|\mathbf{x} - \mathbf{x}'|) \delta(t - t'), & \langle \mathbf{v} \rangle &= 0, \\ \kappa^{ij}(\mathbf{x}) &= \kappa_N(x) \left( \delta^{ij} - \frac{x^i x^j}{x^2} \right) + \kappa_L(x) \frac{x^i x^j}{x^2} + g(x) \epsilon^{ijk} x^k, & x &= |\mathbf{x} - \mathbf{x}'|, \\ \kappa_N(x) &= \kappa_L(x) + x \kappa'_L(x) / 2 \quad \text{for incompressible velocities.} \end{aligned}$$

Equivalently, in the Fourier space we have

$$\langle v^i(\mathbf{k}, t) v^{*j}(\mathbf{k}, t') \rangle = \left[ F(k) \left( \delta^{ij} - \frac{k^i k^j}{k^2} \right) + iG(k) \epsilon^{ijl} k^l \right] \delta(t - t').$$

Two independent functions  $\kappa_L(x)$  and  $g(x)$  or, alternatively, their 3D Fourier transforms  $F(k)$  and  $G(k)$ , describe kinetic energy and kinetic helicity.

Homogeneous isotropic magnetic field correlator is

$$\langle B^i(\mathbf{x}, t) B^j(0, t) \rangle = M_N(x, t) \left( \delta^{ij} - \frac{x^i x^j}{x^2} \right) + M_L(x, t) \frac{x^i x^j}{x^2} + K(x, t) \epsilon^{ijk} x^k,$$

$$M_N(x, t) = M_L(x, t) + (x/2)M'_L(x, t) \quad \text{for solenoidal field.}$$

Equivalently, in Fourier space we have

$$\langle B^i(\mathbf{k}, t) B^{*j}(\mathbf{k}, t) \rangle = F_B(k, t) \left( \delta^{ij} - \frac{k^i k^j}{k^2} \right) - i \frac{H(k, t)}{2k^2} \epsilon^{ijl} k^l.$$

Here  $F_B(k, t)$  is **the magnetic energy spectral function**, and  $H(k, t)$  is the spectral function of **the electric current helicity**.

- Given velocities correlator, the goal is to find  $M_L(x, t)$  and  $K(x, t)$  or, alternatively, their Fourier transforms  $F_B(k, t)$  and  $H(k, t)$ .
- With helicity, need to solve two coupled partial differential equations for magnetic field correlator functions (Vainshtein & Kichatinov 1986).

- Self-adjoint form of the helical dynamo equations is similar to a quantum mechanical “spinor” form with imaginary time (Boldyrev *et al.* 2005):

$$-\partial_t \psi^\alpha = \hat{\mathcal{H}}^{\alpha\beta} \psi^\beta, \quad \alpha = \{1, 2\};$$

- two components of  $\psi^\alpha(x, t)$  are related to  $M_L(x, t)$  and  $K(x, t)$ ;
- Hamiltonian  $\hat{\mathcal{H}}^{\alpha\beta}$  depends on kinetic energy and helicity, and on  $\eta$ ;
- eigenvalues of  $\hat{\mathcal{H}}^{\alpha\beta}$  must be real because it is self-adjoint;
- magnetic field growth rates  $\lambda$  correspond to negative eigenvalues;
- boundary conditions are

$$M_L(0, t) \quad \text{and} \quad K(0, t) \quad = \quad e^{\lambda t} \times \{\text{constant terms}\},$$

$$M_L(x, t) \quad \text{and} \quad K(x, t) \quad \propto \quad e^{\lambda t} \times x^{-2} e^{-k_r x} \times \{\text{oscillatory terms}\},$$

$$\text{where } k_r = \sqrt{\lambda - \lambda_0} / \sqrt{\kappa_L(0) + 2\eta} \quad \text{if } \lambda > \lambda_0 \equiv g^2(0) / [\kappa_L(0) + 2\eta],$$

$$k_r = 0 \quad \text{if } \lambda \leq \lambda_0$$

- Solve equations numerically and find the growth rates  $\lambda$  and the spectra of the magnetic field eigenmodes.

## Results for Kolmogorov Turbulence, $Pr \ll 1$

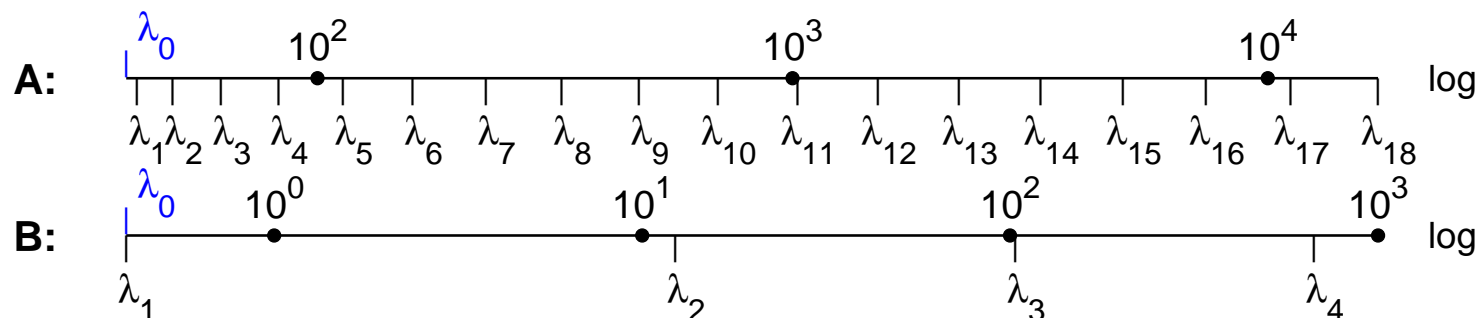
Consider velocity fluctuations with Kolmogorov power spectrum.

$$F(k) = k^{-13/3}, \quad 2 \leq k \leq k_{\max}, \quad -1 \leq h \leq 1,$$

$$G(k) = -hk^{-1}F(k), \quad k_{\max} \approx 2[\kappa_L(0)/\nu]^{3/4} \approx 4\nu^{-3/4}$$

Growth rates  $\lambda_n$  of field eigenmodes ( $\lambda_0$  is the fastest unbound mode)

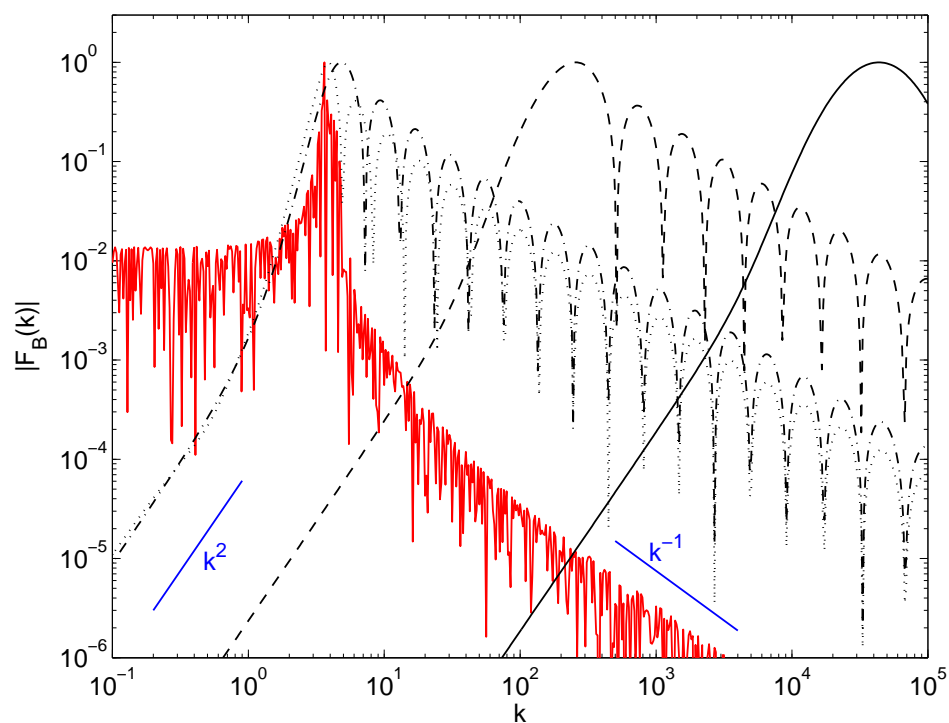
- $k_{\max} = 3 \times 10^7$  and  $Re \approx 10^9 \gg 1$ ;
- $\eta = 10^{-6}$  and  $Rm \approx 10^6 \ll Re$ , therefore,  $Pr \ll 1$ ;
- Kinetic helicity: (A)  $h = 1$ , and (B)  $h = 0.1$ .



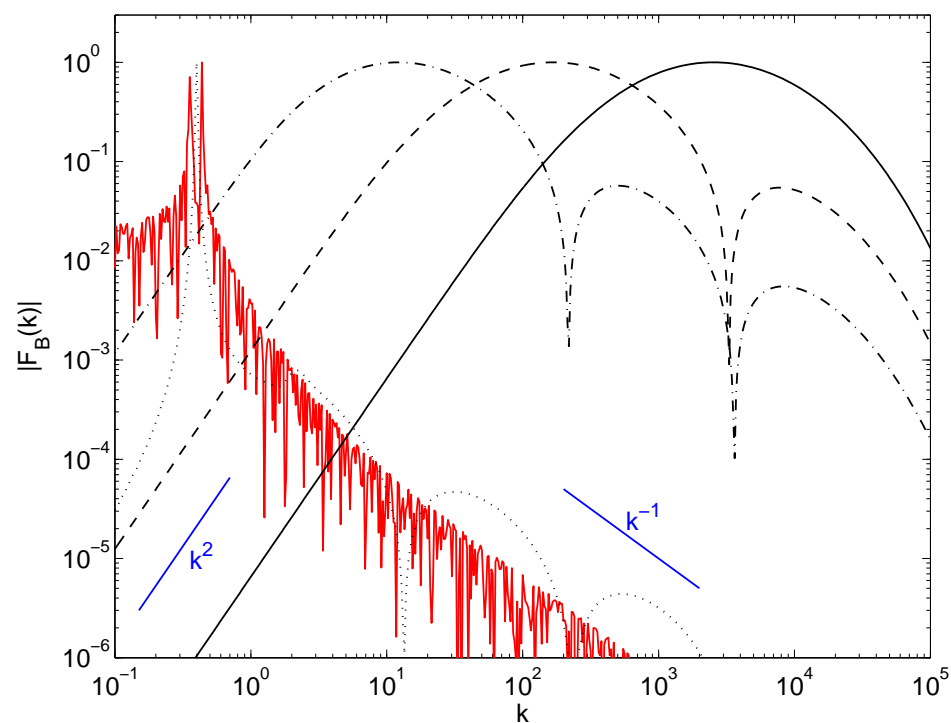
## Magnetic field spectral function $|F_B(k)|$

$\eta = 10^{-6}$ , velocities with Kolmogorov spectrum up to  $k_{\max} = 3 \times 10^7$  ( $\text{Re} \gg 1$ ,  $\text{Pr} \ll 1$ )

Maximal helicity  $h = 1$ . Red line is for fastest growing unbound eigenmode  $\lambda_0 = 39.57$ . Solid, dashed, dash-dot, dotted lines are for bound modes  $\lambda_{18} = 17055 > \lambda_{10} = 695.9 > \lambda_2 = 49.56 > \lambda_1 = 41.73$ .

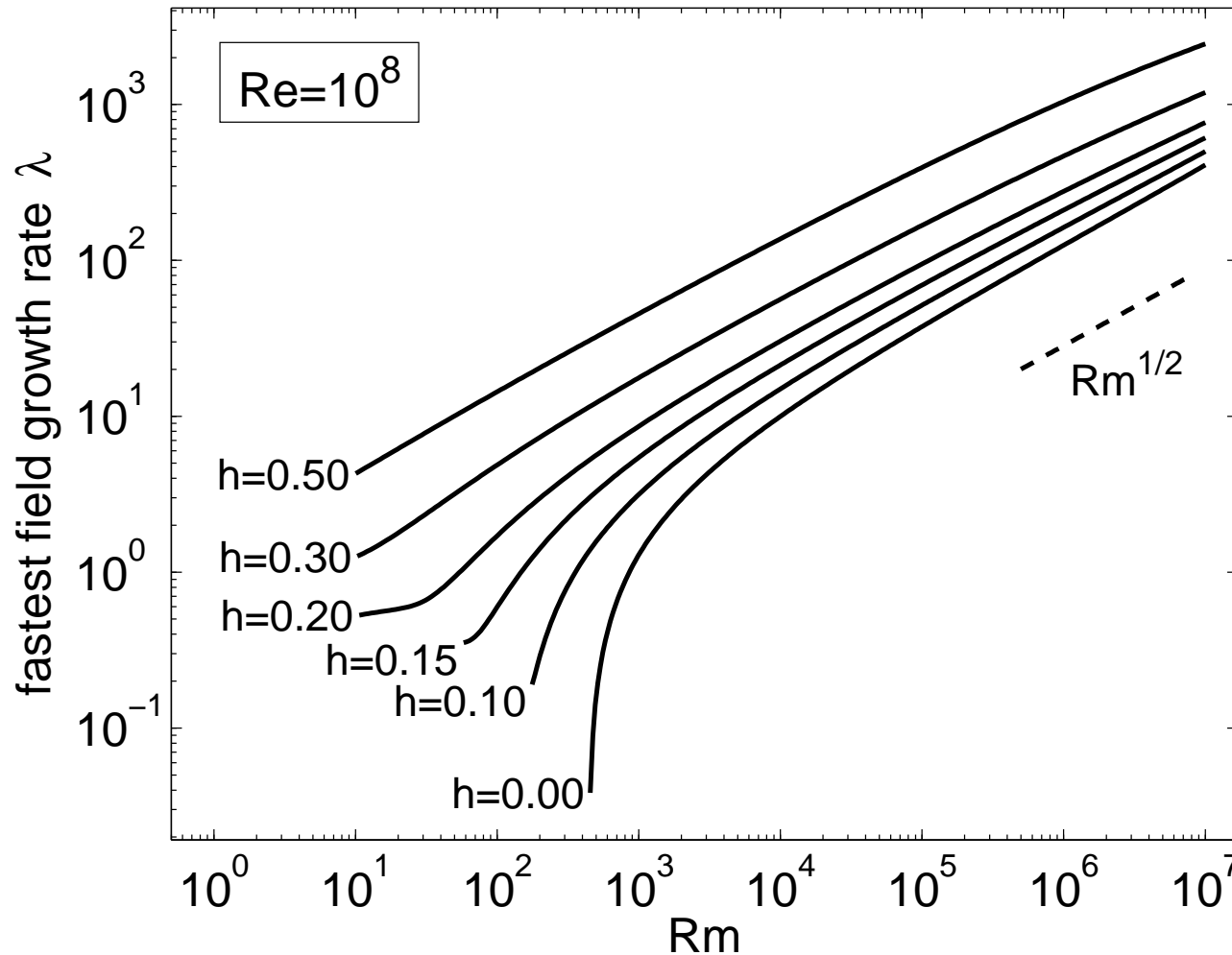


Small helicity  $h = 0.1$ . Red line is for unbound eigenmode  $\lambda_0 = 0.39573$ . Solid, dashed, dash-dot, dotted lines are for bound modes  $\lambda_4 = 669.8 > \lambda_3 = 103.5 > \lambda_2 = 12.30 > \lambda_1 = 0.39581$ .



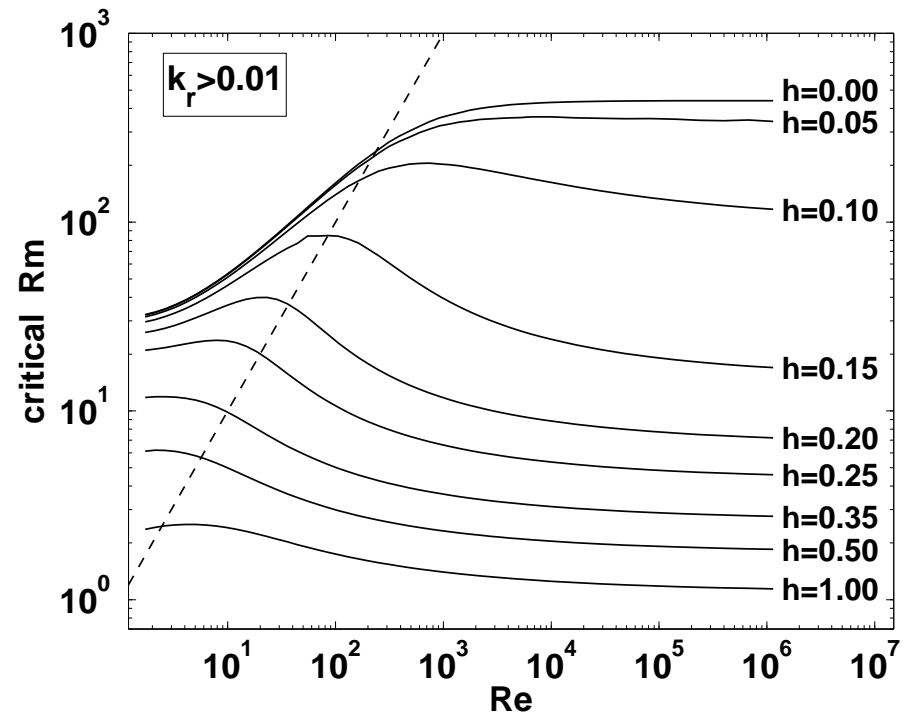
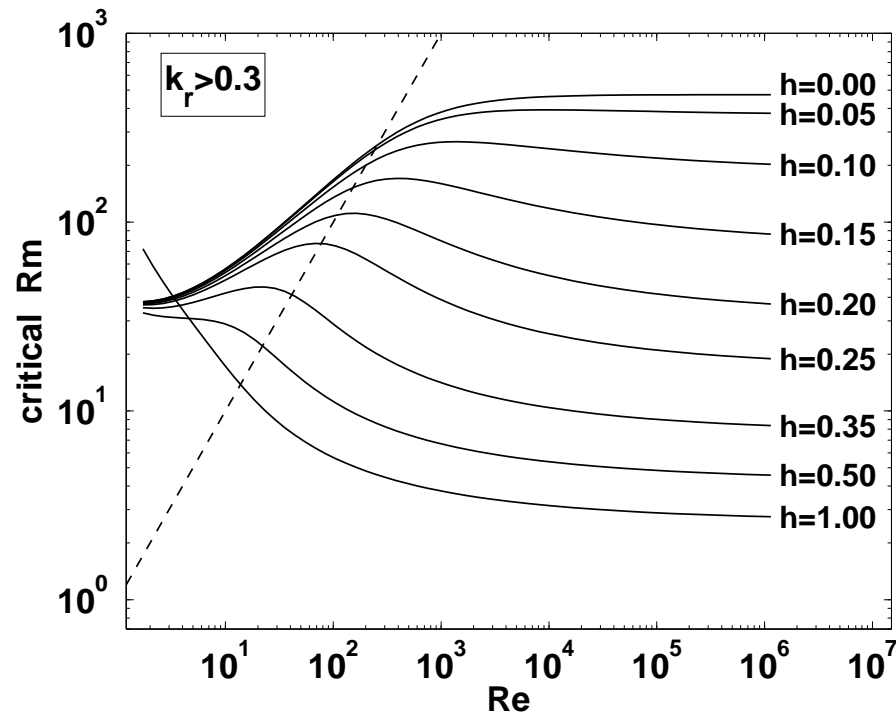
## Growth rate versus $Rm$ (for the case $Pr \ll 1$ )

Theory predicts scaling  $\lambda_{max} \propto Rm^{1/2}$  for the fastest growth rate of the field in the case of non-helical dynamo (Boldyrev and Cattaneo, 2004).



## Critical value of $Rm$

- Dynamo in a finite box differs from that in infinite homogeneous space.
- Consider only bound eigenmodes that can fit into a finite box because they decay exponentially in space:  $M_L(x, t), K(x, t) \propto x^{-2} e^{-k_r x}$ .
- For a box of size  $L$ , a growing bound eigenmode fits in if  $k_r = \sqrt{\lambda - \lambda_0} / \sqrt{\kappa_L(0) + 2\eta} \gtrsim 1/L$ .



## Results and Conclusions

- Conventional alpha-model has limited applicability for large-scale dynamo. This is because
  - in the helical case and when  $Re \gg 1$ , there is always a shallow bound eigenmode  $\lambda_1$  ( $\lambda_1 - \lambda_0 \ll \lambda_0$ ), whose scale is  $\gg$  the velocity scale;
  - both bound and unbound eigenmodes contribute to the large-scale magnetic field growth (even in the kinematic regime);
  - the alpha-mode does not capture any bound eigenmodes, while the shallow bound mode grows faster and dominates the unbound modes.
- Field growth rate is  $\propto Rm^{1/2}$  in the limit  $1 \ll Rm \ll Re$  ( $Pr \ll 1$ ).
- Critical  $Rm$  (when dynamo is exited) is much lower in the case when low-Prandtl turbulence has nonzero kinetic helicity.

**END**

The self-adjoint equations for  $M_L(x, t)$  and  $K(x, t)$  (Boldyrev et al 2005) are

$$M_L = \frac{\sqrt{2}e^{\lambda t}}{x^2} w_2(x), \quad K = -\frac{e^{\lambda t}}{\sqrt{2}x^4} [x^2 w_3(x)]',$$

$$\begin{bmatrix} -\frac{\sqrt{2}}{x} \hat{E} \frac{\sqrt{2}}{x} - \lambda & \frac{\sqrt{2}}{x^3} C \frac{d}{dx} x^2 \\ -x^2 \frac{d}{dx} C \frac{\sqrt{2}}{x^3} & x^2 \frac{d}{dx} \frac{B}{x^4} \frac{d}{dx} x^2 - \lambda \end{bmatrix} \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} = 0,$$

$$\hat{E} = -\frac{1}{2} x \frac{d}{dx} B \frac{d}{dx} x + \frac{1}{\sqrt{2}} (A - xA'),$$

$$A(x) = \sqrt{2} [2\eta + \kappa_N(0) - \kappa_N(x)],$$

$$B(x) = 2\eta + \kappa_L(0) - \kappa_L(x),$$

$$C(x) = \sqrt{2} [g(0) - g(x)] x.$$

- The above equations describe the magnetic field growth in the Kazantsev-Kraichnan model. These equations were numerically solved by using the 4<sup>th</sup>-order Runge-Kutta numerical integration method.
- The growth rates  $\lambda$  and spectra of the magnetic field eigenmodes were found (note that all  $\lambda$ -s must be real because equations are self-adjoint).

## Results for Kolmogorov Turbulence (all results)

Consider velocity fluctuations with Kolmogorov power spectrum.

$$F(k) = k^{-13/3}, \quad 2 \leq k \leq k_{\max}, \quad -1 \leq h \leq 1,$$

$$G(k) = -hk^{-1}F(k), \quad k_{\max} \approx 2[\kappa_L(0)/\nu]^{3/4} \approx 4\nu^{-3/4}$$

Growth rates  $\lambda_n$  of field eigenmodes,  $\eta = 10^{-6}$ ,  $\lambda_0$  is the fastest unbound mode

A & B:  $h = 1$  &  $0.1$ ,

$k_{\max} = 3$ ,

$\text{Re} \sim 1$ ,

$\text{Pr} \gg 1$ .

C & D:  $h = 1$  &  $0.1$ ,

$k_{\max} = 3000$ ,

$\text{Re} \gg 1$ ,

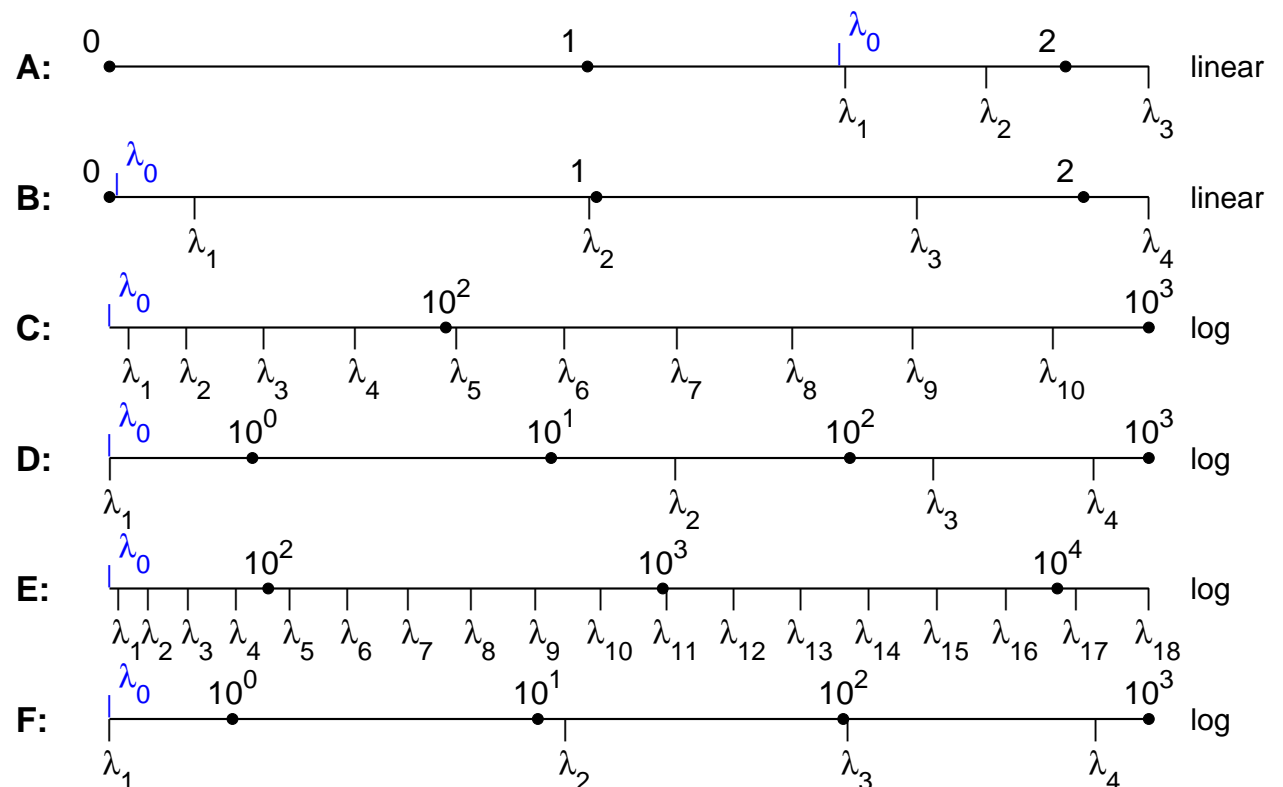
$\text{Pr} \gg 1$ .

E & F:  $h = 1$  &  $0.1$ ,

$k_{\max} = 3 \times 10^7$ ,

$\text{Re} \gg 1$ ,

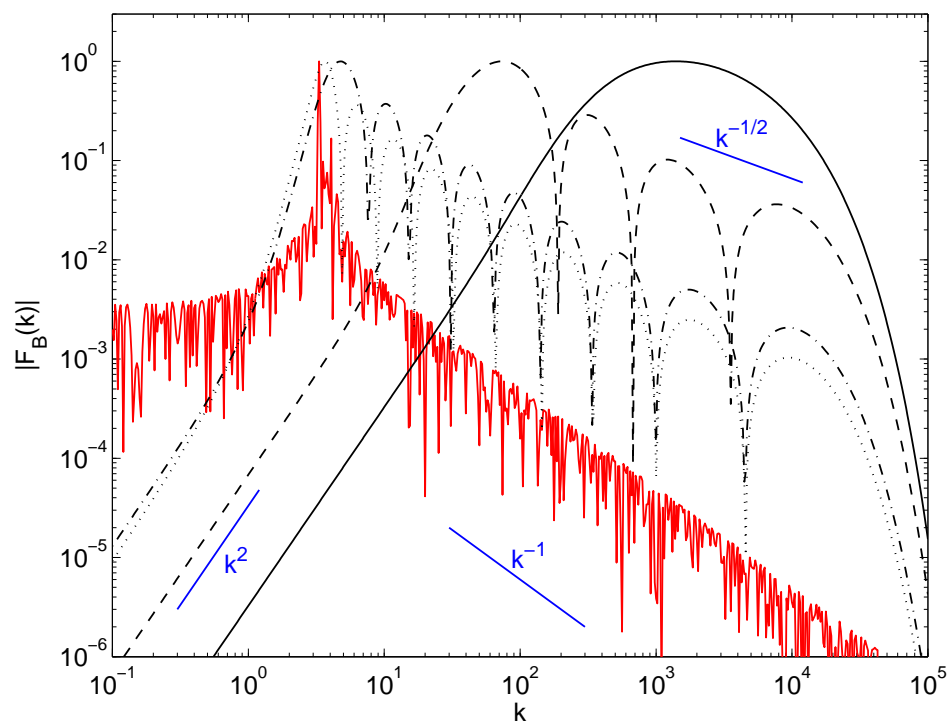
$\text{Pr} \ll 1$ .



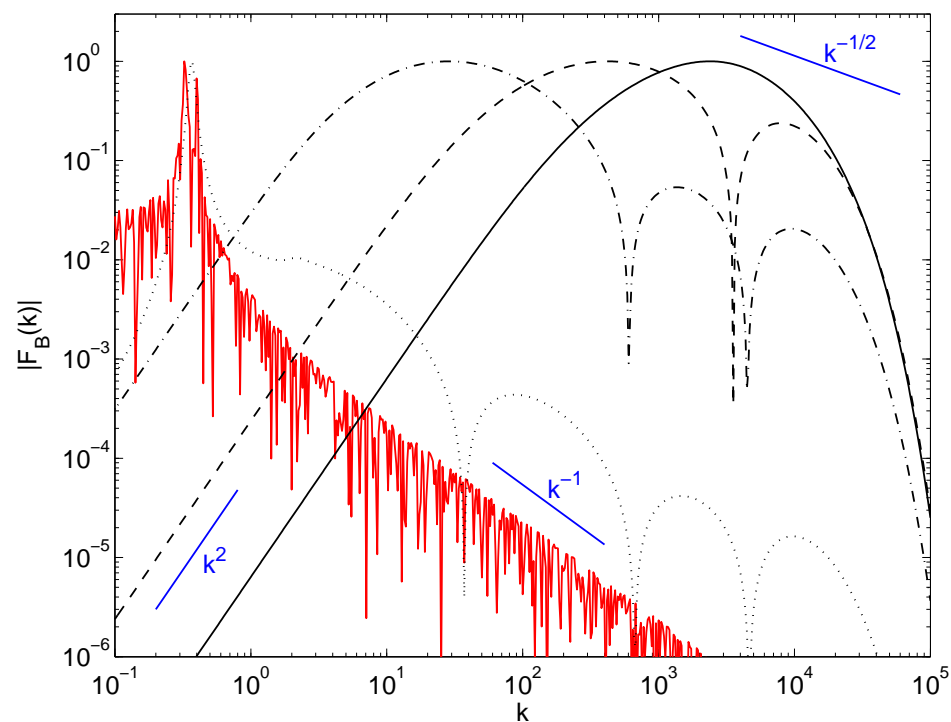
Plots of absolute value of the magnetic field spectral function  $F_B(k)$ 

$\eta = 10^{-6}$ , velocities with Kolmogorov spectrum up to  $k_{\max} = 3000$  ( $\text{Re} \gg 1, \text{Pr} \gg 1$ )

Maximal helicity  $h = 1$ . Red line is for fastest growing unbound eigenmode  $\lambda_0 = 33.23$ . Solid, dashed, dash-dot, dotted lines are for bound modes  $\lambda_{10} = 731.1 > \lambda_7 = 213.4 > \lambda_2 = 42.74 > \lambda_1 = 35.41$ .



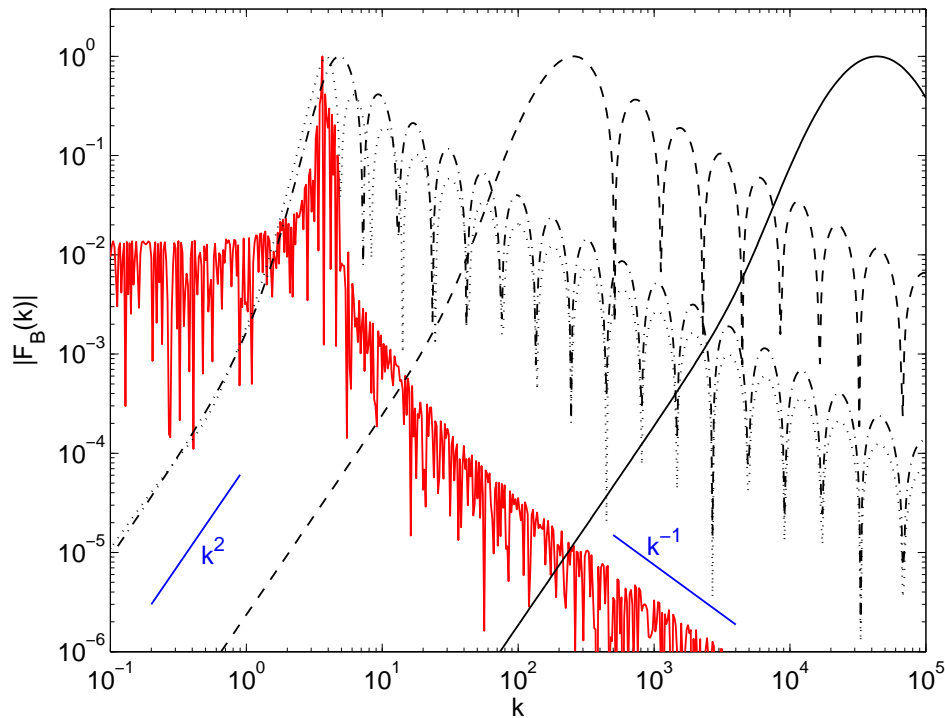
Small helicity  $h = 0.1$ . Red line is for unbound eigenmode  $\lambda_0 = 0.3323$ . Solid, dashed, dash-dot, dotted lines are for bound eigenmodes  $\lambda_4 = 654.2 > \lambda_3 = 190.1 > \lambda_2 = 26.03 > \lambda_1 = 0.3336$ .



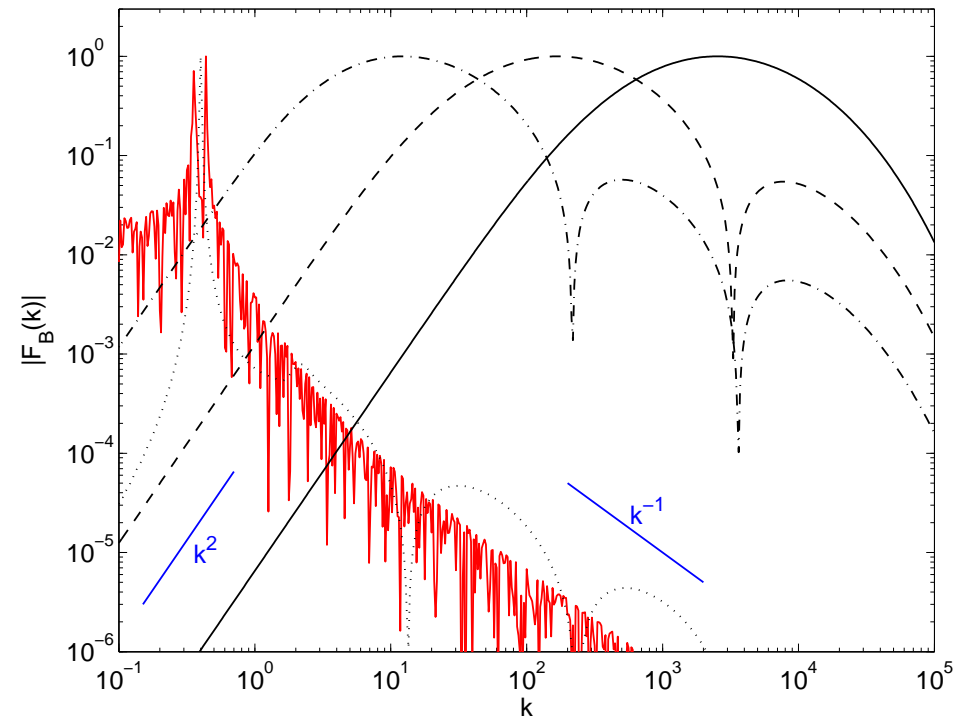
Plots of absolute value of the magnetic field spectral function  $F_B(k)$ 

$\eta = 10^{-6}$ , velocities with Kolmogorov spectrum up to  $k_{\max} = 3 \times 10^7$  ( $\text{Re} \gg 1, \text{Pr} \ll 1$ )

Maximal helicity  $h = 1$ . Red line is for fastest growing unbound eigenmode  $\lambda_0 = 39.57$ . Solid, dashed, dash-dot, dotted lines are for bound modes  $\lambda_{18} = 17055 > \lambda_{10} = 695.9 > \lambda_2 = 49.56 > \lambda_1 = 41.73$ .



Small helicity  $h = 0.1$ . Red line is for unbound eigenmode  $\lambda_0 = 0.39573$ . Solid, dashed, dash-dot, dotted lines are for bound modes  $\lambda_4 = 669.8 > \lambda_3 = 103.5 > \lambda_2 = 12.30 > \lambda_1 = 0.39581$ .



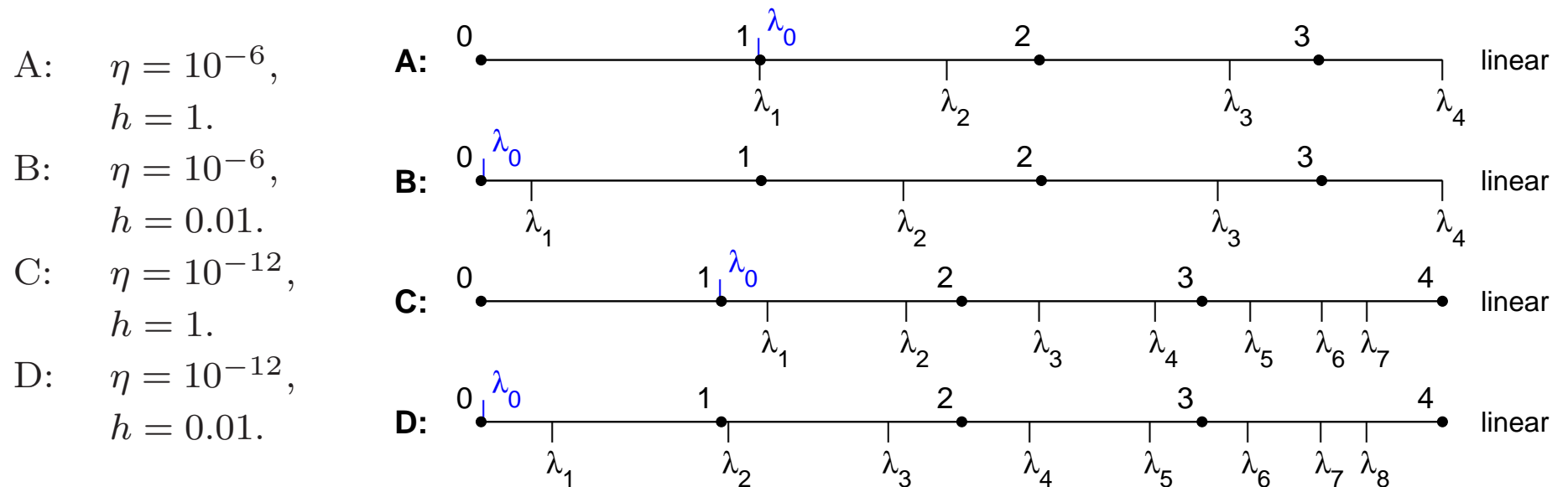
## Results for Gaussian Turbulent Velocity Field

Consider Gaussian functions of the velocity correlation tensor (all velocity fluctuations are on scale of order unity,  $\text{Re} \sim 1$ ),

$$\kappa_L(x) = e^{-x^2}, \quad g(x) = \frac{4h\sqrt{2e}}{27\sqrt{3}} \left( 5 - \frac{4x^2}{3} \right) e^{-2x^2/3}, \quad -1 \leq h \leq 1.$$

Growth rates  $\lambda_n$  of field eigenmodes,  $\lambda_0$  is the fastest unbound mode

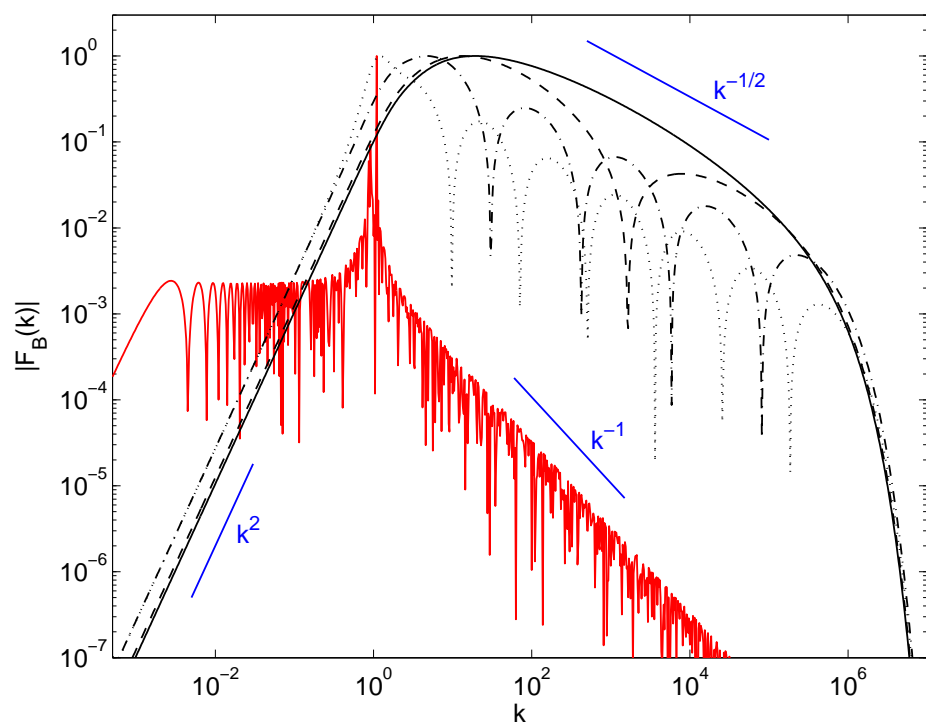
$\text{Re} \sim 1, \text{Pr} \sim 1$



Plots of absolute value of the magnetic field spectral function  $F_B(k)$ 

$\eta = 10^{-12}$  and single-scale Gaussian velocities ( $\text{Re} \sim 1$ ,  $\text{Pr} \gg 1$ )

Maximal helicity  $h = 1$ . Red line is for fastest growing unbound eigenmode  $\lambda_0 = 0.9943$ . Solid, dashed, dash-dot, dotted lines are for bound modes  $\lambda_7 = 3.686 > \lambda_6 = 3.499 > \lambda_3 = 2.322 > \lambda_1 = 1.192$ .



Small helicity  $h = 0.01$ . Red line is for unbound eigenmode  $\lambda_0 = 9.943 \times 10^{-5}$ . Solid, dashed, dash-dot, dotted lines are for bound eigenmodes  $\lambda_8 = 3.685 > \lambda_7 = 3.494 > \lambda_3 = 1.694 > \lambda_1 = 0.2948$ .

