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Negative energy waves and MHD stability of rotating plasma

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Outline

- Introduction
- Energy of eigenmodes in ideal MHD
- Magnetorotational instability
- Modified energy principle
- Summary

Introduction

- Energy consideration is of primary significance in MHD stability analysis
- Energy principle ($\delta^2 W>0$ for stability):

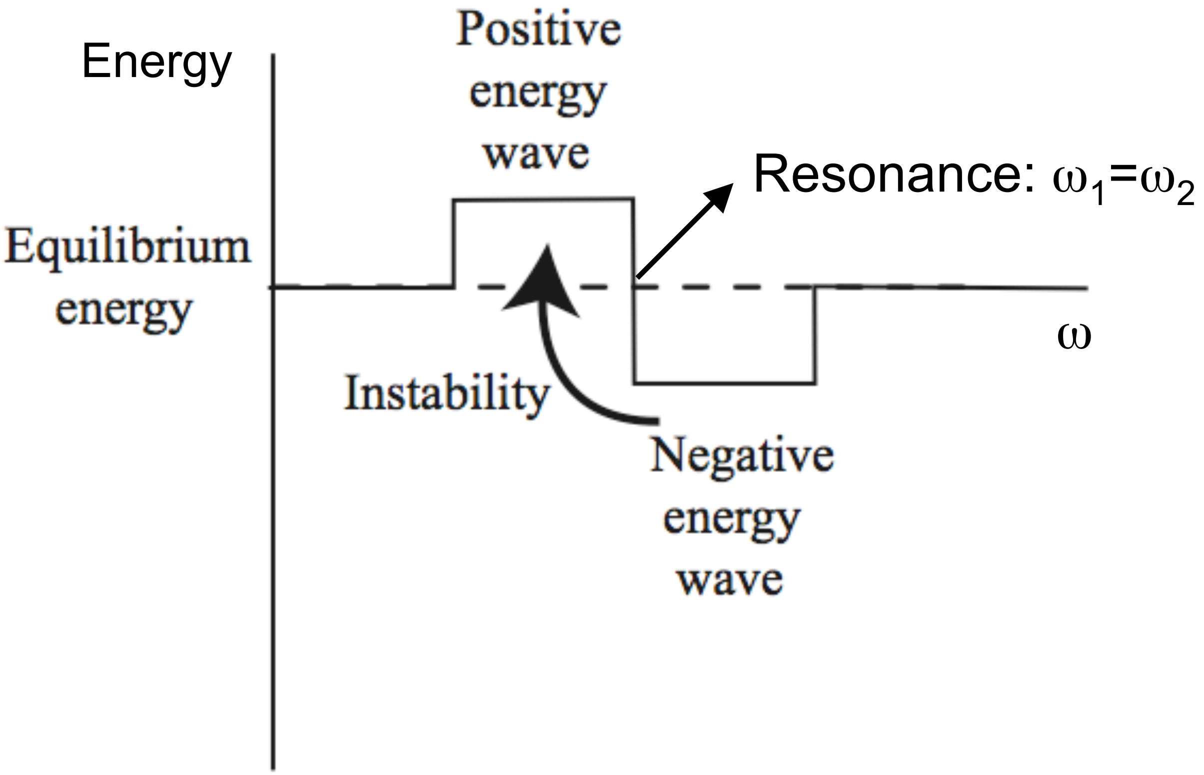
  - stability criterion for static systems (without flows)

  - only sufficient stability condition in moving media since waves with negative energy ($\delta^2 W<0$) can be excited
- Negative energy waves (NEWs) are potential source of instability. Classical examples:

  - hydrodynamical Kelvin-Helmholtz instability

  - plasma-beam instability
- Modified energy principle should be used to analyze stability of MHD systems with flows

Positive and negative energy waves



- Excitation of NEW leads to decrease of total system energy
- PEW and NEW with frequencies ω_1 and ω_2 can coexist in the system
- When $\omega_1=\omega_2$ (resonance) energy can be transferred from NEW to PEW leading to instability

Ideal MHD

$$\begin{aligned}
\dot{\rho} &= -\nabla \cdot (\rho \mathbf{V}) \\
\rho \dot{\mathbf{V}} &= -\rho(\mathbf{V} \cdot \nabla)\mathbf{V} - \nabla P + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} \\
\dot{P} &= -\mathbf{V} \cdot \nabla P - \gamma P \nabla \cdot \mathbf{V} \\
\dot{\mathbf{B}} &= \nabla \times (\mathbf{V} \times \mathbf{B}), \quad \nabla \cdot \mathbf{B} = 0
\end{aligned}$$

Perturbations

$$\begin{aligned}
\delta \mathbf{V} &= \dot{\boldsymbol{\xi}} + (\mathbf{V} \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\mathbf{V} \\
\delta \rho &= -\nabla \cdot (\rho \boldsymbol{\xi}) \\
\delta P &= -\boldsymbol{\xi} \cdot \nabla P - \gamma P \nabla \cdot \boldsymbol{\xi} \\
\delta \mathbf{B} &= \nabla \times (\boldsymbol{\xi} \times \mathbf{B})
\end{aligned}$$

- ξ is a Lagrangian displacement vector

- Linearized equation of dynamics for ξ:

$$\rho\ddot{\boldsymbol{\xi}} + 2\rho(\mathbf{V}\cdot\nabla)\dot{\boldsymbol{\xi}} - \mathbf{F}(\boldsymbol{\xi}) = 0$$

- Force operator:

$$\begin{aligned}\mathbf{F}(\boldsymbol{\xi}) = & -\rho(\mathbf{V}\cdot\nabla)^2\boldsymbol{\xi} + \rho(\boldsymbol{\xi}\cdot\nabla)(\mathbf{V}\cdot\nabla)\mathbf{V} + \nabla\cdot(\rho\boldsymbol{\xi})(\mathbf{V}\cdot\nabla)\mathbf{V} \\ & - \nabla\delta P + \frac{1}{4\pi}(\nabla\times\delta\mathbf{B})\times\mathbf{B} + \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\delta\mathbf{B}.\end{aligned}$$

- Properties:

$$\int \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})\,d^3\mathbf{r} = \int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\eta})\,d^3\mathbf{r},$$

$$\int \rho\boldsymbol{\eta}\cdot(\mathbf{V}\cdot\nabla)\boldsymbol{\xi}\,d^3\mathbf{r} = -\int \rho\boldsymbol{\xi}\cdot(\mathbf{V}\cdot\nabla)\boldsymbol{\eta}\,d^3\mathbf{r}.$$

Energy

$$E = \frac{1}{2}\int \left(\rho|\dot{\boldsymbol{\xi}}|^2 - \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\right) d^3\mathbf{r}$$

Kinetic energy

Potential energy

- Energy is conserved: d*E*/d*t* = 0

Eigenvalue problem

$$\boldsymbol{\xi}(\mathbf{r}, t) = \hat{\boldsymbol{\xi}}(\mathbf{r})e^{-i\omega t}$$

$$\omega^2\rho\hat{\boldsymbol{\xi}} + 2i\omega\rho(\mathbf{V}\cdot\nabla)\hat{\boldsymbol{\xi}} + \mathbf{F}(\hat{\boldsymbol{\xi}}) = 0$$

Energy of eigenmode

$$A\,\omega^2 - 2B\,\omega - C = 0 \qquad \omega = \frac{B + s\sqrt{B^2 + AC}}{A}$$

where either $s = 1$ or $s = -1$ for a particular eigenmode.

$$A = \int \rho|\hat{\boldsymbol{\xi}}|^2\, d^3\mathbf{r},\; B = -\,i\int \rho\hat{\boldsymbol{\xi}}^* \cdot (\mathbf{V}\cdot\nabla)\hat{\boldsymbol{\xi}}\, d^3\mathbf{r} \;\; C = -\int \hat{\boldsymbol{\xi}}^* \cdot \mathbf{F}(\hat{\boldsymbol{\xi}})\, d^3\mathbf{r}$$

- Energy: $E = \frac{1}{2}(A\,|\,\omega|^2 + C)e^{2\gamma t}$ $\qquad \gamma = \mathrm{Im}(\omega)$
- Unstable eigenmode: $B^2 + AC < 0$

$$E = 0$$

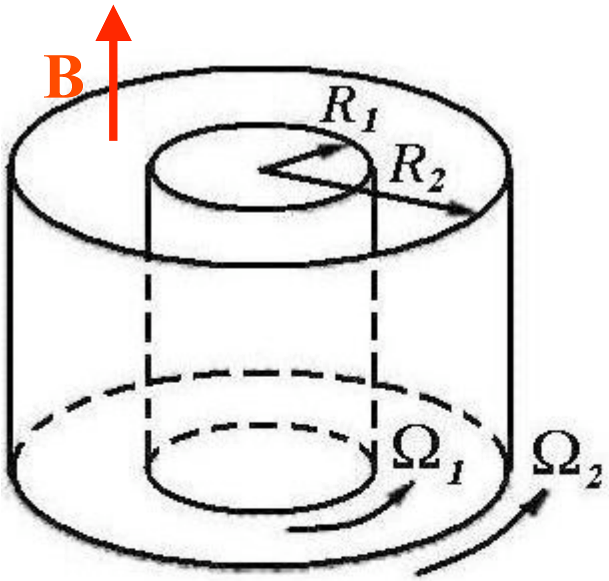
- Stable eigenmode: $B^2 + AC > 0$

$$E = s\omega\,\sqrt{B^2 + AC}$$

- Energy of stable eigenmode can be either positive (PEW) or negative (NEW)

Magnetorotational instability

- MRI is instability of conducting fluid rotating in transverse magnetic field



- Equilibrium:

$$\mathbf{B} = B_0\mathbf{e}_z. \quad \mathbf{V} = r\Omega(r)\mathbf{e}_\varphi, \quad \Omega(r) = \frac{\Omega_1 r_1^2}{r^2}$$

- Eigenmode representation:

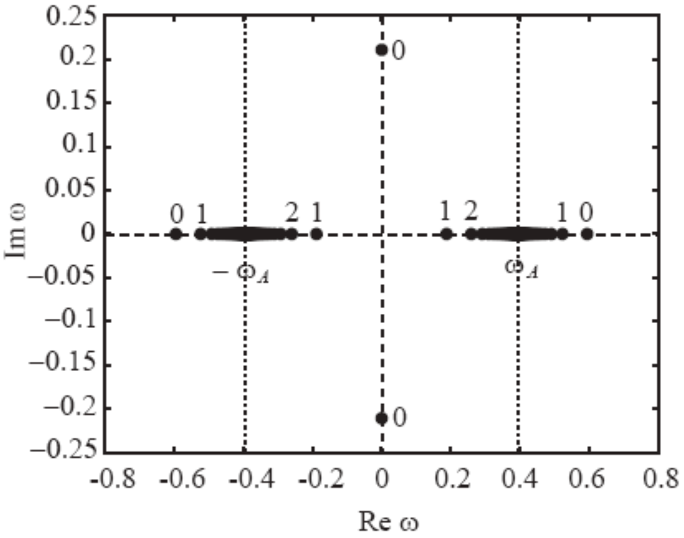
$$\boldsymbol{\xi}(\mathbf{r}, t) = \boldsymbol{\xi}(r)\exp\left(-i\omega t + im\phi + ik_z z\right).$$

- Alfven frequency:

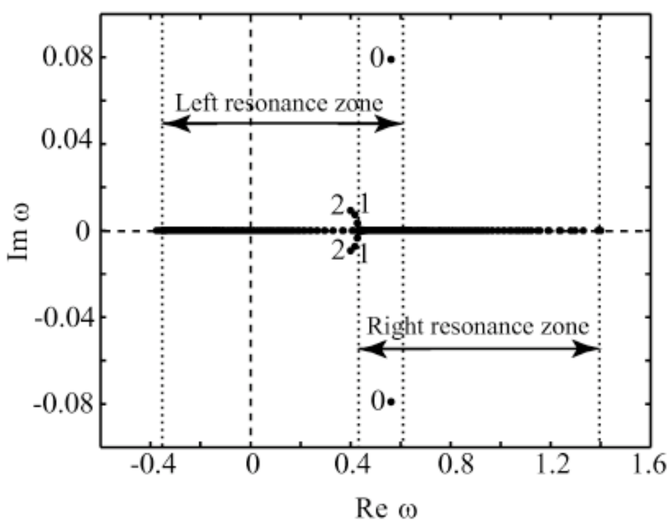
$$\omega_A = \frac{k_z B_0}{\sqrt{4\pi\rho}}.$$

MRI eigen-spectrum

m=0



m=1

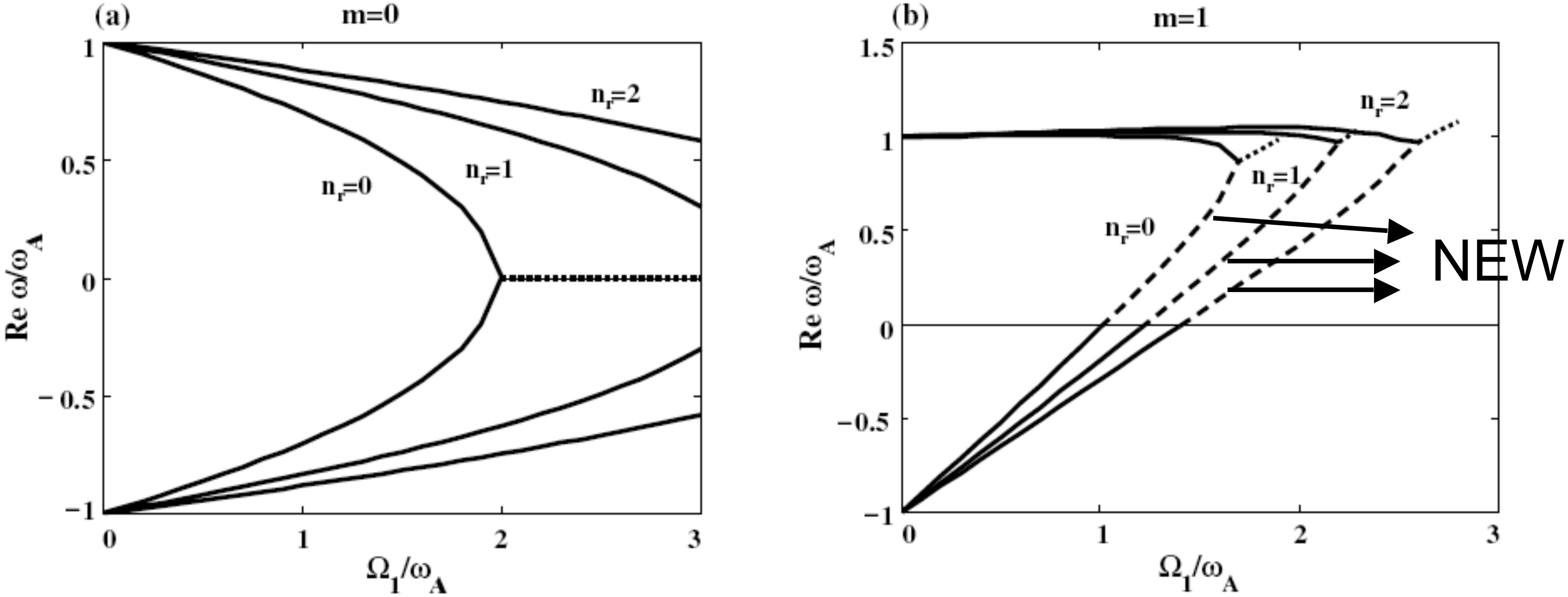


- Two types of eigenmodes are possible for general MHD systems with flows:

  - Symmetric (spectrum is symmetric about the origin)

  - Non-symmetric (spectrum is not symmetric about the imaginary axis)

Spectrum dependence on Ω_1/ω_A



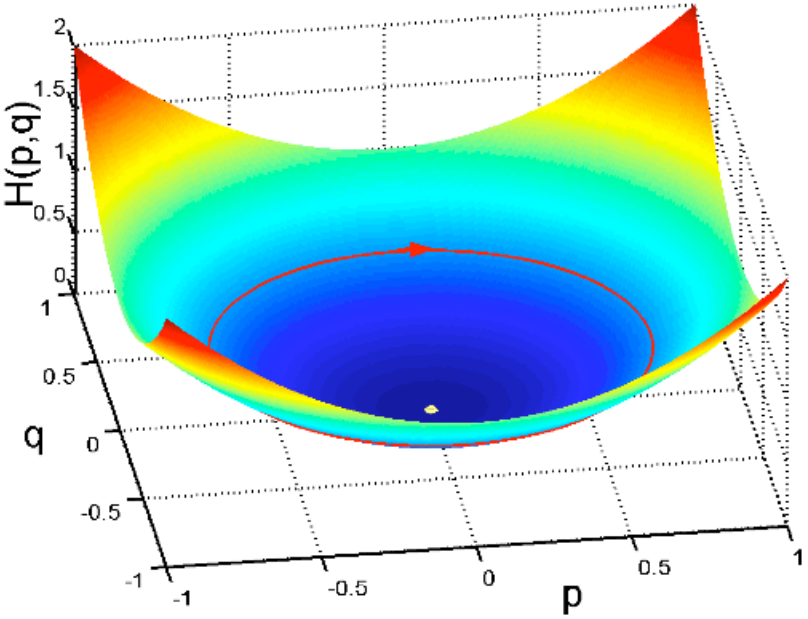
NEW

- Energy of symmetric modes is never negative. Energy principle can be applied.
- Energy of stable non-symmetric modes can be negative. Modified energy principle should be used for their stability analysis

What is the energy principle?

- Lyapunov theorem (sufficient stability condition):

  If there is an integral of motion (conserved quantity) which has a local minimum at the equilibrium state then the equilibrium is stable
- No general method to construct Lyapunov functional
- For conservative systems energy is a Lyapunov functional candidate.
- Example: oscillator with Hamiltonian (energy) $H=p^2+q^2$ is stable



Energy principle in ideal MHD

- Energy of perturbations (second variation of total energy):

$$E = \frac{1}{2}\int \left(\rho|\dot{\boldsymbol{\xi}}|^2 - \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\right) d^3\mathbf{r}$$

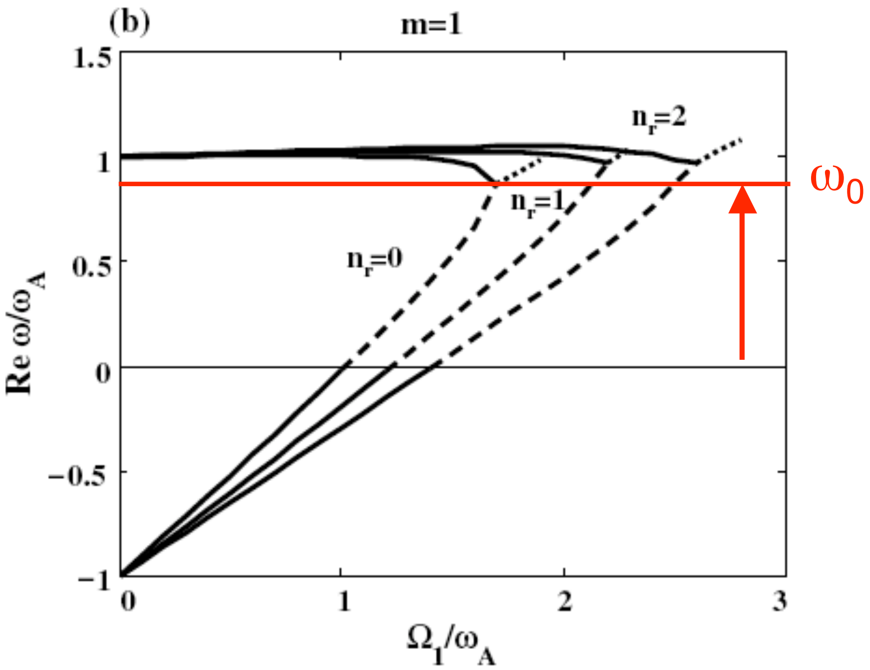
- Energy principle (from Lyapunov theorem):

  if potential energy $W(\boldsymbol{\xi}) = -\frac{1}{2}\int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi}) d^3\mathbf{r} \geq 0$

  is positive for any displacement ***ξ*** then equilibrium is stable

- When **V** = 0 (no equilibrium flow) this condition is criterion (both sufficient and necessary) for stability
- When **V** $\neq 0$ this is only sufficient condition for stability.

Modified energy principle for MRI



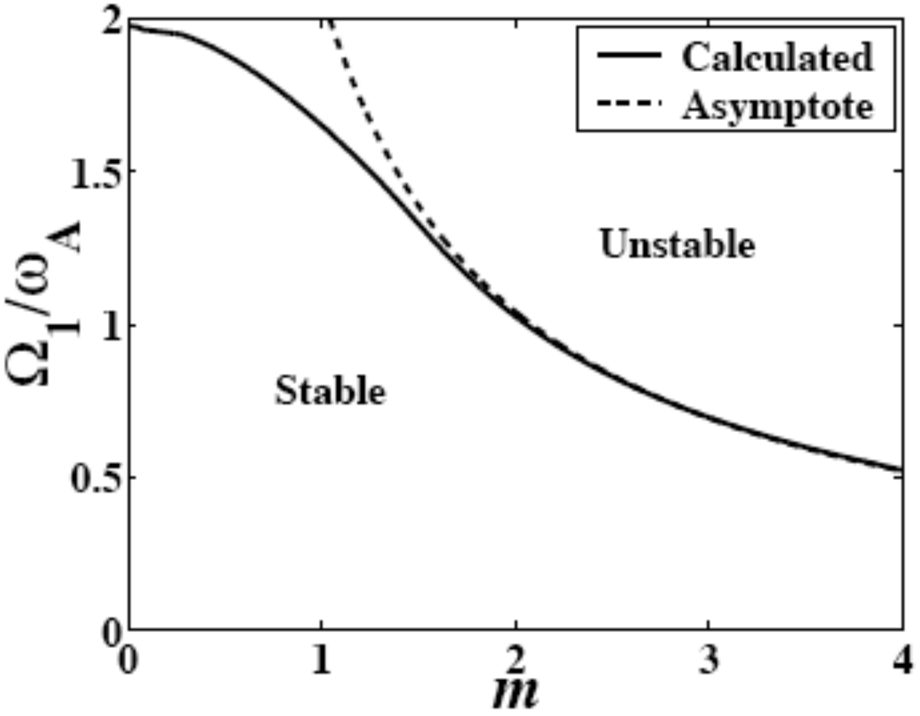
- Transition to reference frame rotating with frequency ω_0 eliminates all NEWs in stable system

- Then the usual energy principle applies: system is stable if potential energy (in moving reference frame) is positive for all perturbations

$$\tilde{W}(\boldsymbol{\xi}) = -\frac{1}{2}\int \boldsymbol{\xi}^* \cdot \tilde{\mathbf{F}}(\boldsymbol{\xi})\, d^3\mathbf{r} =$$

$$= -\frac{1}{2}\int \Big(\omega_0^2|\boldsymbol{\xi}|^2 + 2i\omega_0\rho\boldsymbol{\xi}^* \cdot (\mathbf{V}\cdot\nabla)\boldsymbol{\xi} + \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\Big) d^3\mathbf{r}.$$

MRI threshold



- Modified energy principle gives the asymptote of MRI threshold (dashed line): $$\frac{\Omega_1}{\omega_A} = \frac{2}{m\left(1-(R_1/R_2)^2\right)}$$
- Such simple modification of energy principle does not work for all MHD systems with flows.

Summary

- The physical difference between symmetric and non-symmetric modes was demonstrated
- Resonance of positive and negative energy waves is a universal mechanism for any non-symmetric instability
- Modified energy principle was developed for stability study of systems with MHD flows
- Under certain assumptions this method gives both sufficient and necessary stability condition

New invariants in ideal MHD

- Energy principle for the case $\mathbf{V} \neq \mathbf{0}$ can be improved by inclusion of new invariants (conservation laws).
- Energy: $E = \frac{1}{2}\int \left(\rho|\dot{\boldsymbol{\xi}}|^2 - \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\right) d^3\mathbf{r}$
- New set of invariants (*n* is *n*-th time derivative):

$$E_n = \frac{1}{2}\int \left(\rho|\boldsymbol{\xi}^{(n+1)}|^2 - \boldsymbol{\xi}^{*(n)} \cdot \mathbf{F}(\boldsymbol{\xi}^{(n)})\right) d^3\mathbf{r},$$

- No nonlinear analogues!
- Higher time derivatives of displacement vector $\boldsymbol{\xi}^{(n)}$ are expressed via lower derivatives:
- For example: $\boldsymbol{\xi}^{(n+2)} = -2(\mathbf{V}\cdot\nabla)\boldsymbol{\xi}^{(n+1)} + \frac{\mathbf{F}(\boldsymbol{\xi}^{(n)})}{\rho}$

$$E_1(\dot{\boldsymbol{\xi}}, \boldsymbol{\xi}) = \frac{1}{2}\int \left(\frac{1}{\rho}\left|\mathbf{F}(\boldsymbol{\xi}) - 2\rho(\mathbf{V}\cdot\nabla)\dot{\boldsymbol{\xi}}\right|^2 - \dot{\boldsymbol{\xi}}^* \cdot \mathbf{F}(\dot{\boldsymbol{\xi}})\right) d^3\mathbf{r}$$

Improved energy principle

- Lyapunov functional candidate:

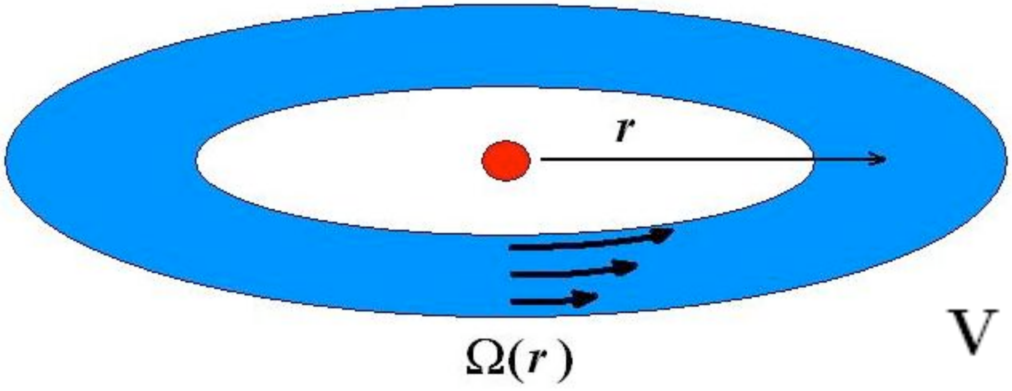
$$U(\dot{\xi}, \xi) = \sum_{n=0}^{N} \lambda_n E_n(\dot{\xi}, \xi)$$

- Improved energy principle:

  If there exist Lagrange multipliers λ_n such that *U* is positively-definite for all displacements then equilibrium is stable

- For the systems with countable set of eigenmodes this gives also a necessary condition for stability

Rotation of fluid in gravitational field



Equilibrium
(no m. field):

$$\mathbf{V} = r\Omega(r)\mathbf{e}_\varphi, \quad r\Omega^2(r) = \Phi'$$

- Eigenmode stability analysis

$$\boldsymbol{\xi}(t,\mathbf{r}) = \sum_{m,k_z} \xi_{m,k_z}(t,r)\exp\{im(\varphi-\Omega(r)t)+ik_z z\}$$

$$\ddot{\boldsymbol{\xi}} + 2\Omega\hat{\mathbf{A}}\,\dot{\boldsymbol{\xi}} - \hat{\mathbf{B}}\,\boldsymbol{\xi} = 0$$

$$\hat{\mathbf{A}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\mathbf{B}} = \begin{pmatrix} -r(\Omega^2)' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Rayleigh criterion

- Spectral method gives:

$$4\Omega^2 + r(\Omega^2)' \geq 0$$

- The first two integrals:

$$E_0 = \frac{1}{2}\Big(|\dot{\boldsymbol{\xi}}|^2 - \boldsymbol{\xi}^{*T}\hat{\mathrm{B}}\boldsymbol{\xi}\Big) = \frac{1}{2}\Big(|\dot{\xi}_r|^2 + |\dot{\xi}_\varphi|^2 + |\dot{\xi}_z|^2 + r(\Omega^2)'|\xi_r|^2\Big)$$

$$\begin{aligned} E_1 &= \frac{1}{2}\Big(|\hat{\mathrm{B}}\boldsymbol{\xi} - 2\Omega\hat{\mathrm{A}}\dot{\boldsymbol{\xi}}|^2 - \dot{\boldsymbol{\xi}}^{*T}\hat{\mathrm{B}}\dot{\boldsymbol{\xi}}\Big) = \\ &= \frac{1}{2}\Big(\Big|r(\Omega^2)'\,\xi_r - 2\Omega\dot{\xi}_\varphi\Big|^2 + \Big(4\Omega^2 + r(\Omega^2)'\Big)|\dot{\xi}_r|^2\Big). \end{aligned}$$

- If we choose U=E_1 we arrive at the same stability criterion